Math G2 Practices 12

Multi-variable integration

1 Parametrization

1. What is the domain described by the inequalities $-\sqrt{1-y^2} \le x \le 1-y$, $0 \le y \le 1$?

   Solution: First let us consider the inequality involving constants: $0 \le y \le 1$, so we only have to consider these values of y.

   Then, let us observe the lower bound for x, meaning that $-\sqrt{1-y^2} = x$. This means that $x^2 + y^2 = 1$, but we know that $0 \le y \le 1$ and also $x \le 0$ (since the root is positive). So this is a section of the unit circle which is in the second quadrant.

   Now let us examine the upper bound: $x = 1 - y$, which is a straight line. See Figure 1 for the shape of the domain.

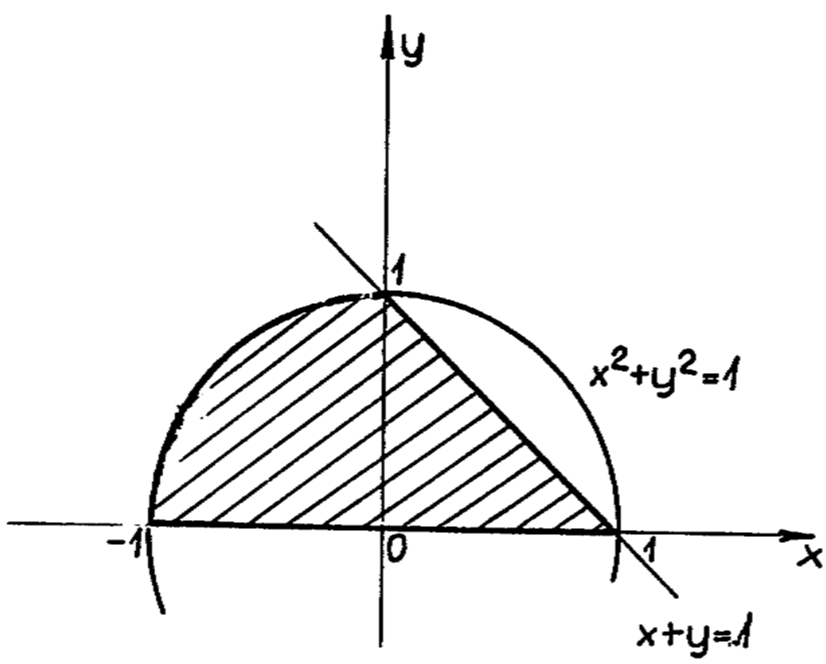

Figure 1: The solution of Exercise 1.

2. What is the domain described by the inequalities $0 \le r \le 2\sqrt{\cos(2\varphi)}$ and $-\dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{4}$?

   Solution: Since $-\dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{4}$, we are between the lines $y = x$ and $y = -x$. The $r = 0$ bound means the origin, and the $r = 2\sqrt{\cos(2\varphi)}$ is an equation of a lemniscate. The shape can be seen in Figure 2.

3. What is the domain described by the inequalities $0 \le z \le 1 - 9x^2 - 4y^2$, $-\dfrac{1}{2}\sqrt{1-9x^2} \le y \le \dfrac{1}{2}\sqrt{1-9x^2}$, and $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$?

   Solution: First let us observe that we are between the values $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$.

   Then, the borders for y are $y = -\dfrac{1}{2}\sqrt{1-9x^2}$ and $y = \dfrac{1}{2}\sqrt{1-9x^2}$, which mean an ellipse $\dfrac{x^2}{\left(\frac{1}{3}\right)^2} + \dfrac{y^2}{\left(\frac{1}{2}\right)^2}$ on the $x-y$ plane: this ellipse is between $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$ and also $-\dfrac{1}{2} \le y \le \dfrac{1}{2}$.

   The $z = 0$ lower border means the $x-y$ plane itself, and the other $z = 1 - 9x^2 - 4y^2$ is a paraboloid. The final shape can be seen in Figure 3.

4. Describe the domain given by Figure 4!

   Solution: Let us use polar coordinates! Then $0 \le \varphi \le \frac{\pi}{4}$ and $0 \le r \le e^{\varphi}$.

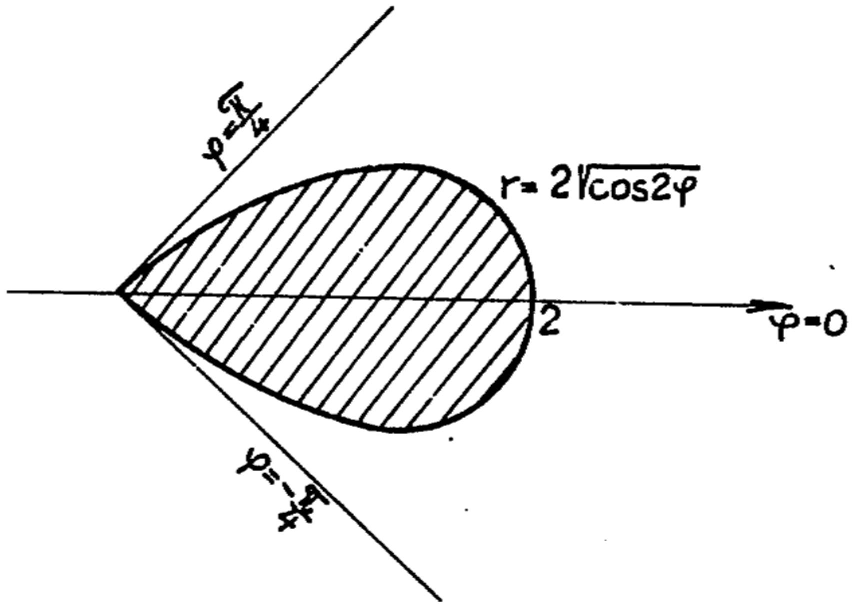

Figure 2: The solution of Exercise 2.

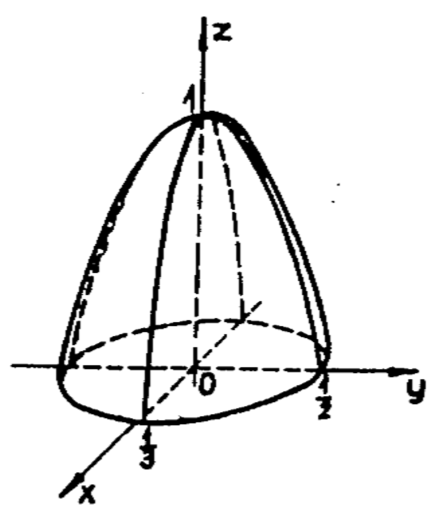

Figure 3: The solution of Exercise 3.

5. Describe the domain given by Figure 5!

   Solution: Let us use cylindrical coordinates! Then in this case $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ and $z = z$. It is clear that $\frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}$. Also, the distance from the z axis is at most two, so $0 \leq r \leq 2$. Moreover, the values of z are below the surface $z = x^2 + y^2$, meaning that

$$0 \leq z \leq x^2 + y^2 = r^2(\cos^2(\varphi) - \sin^2(\varphi)) = r^2\cos(2\varphi).$$

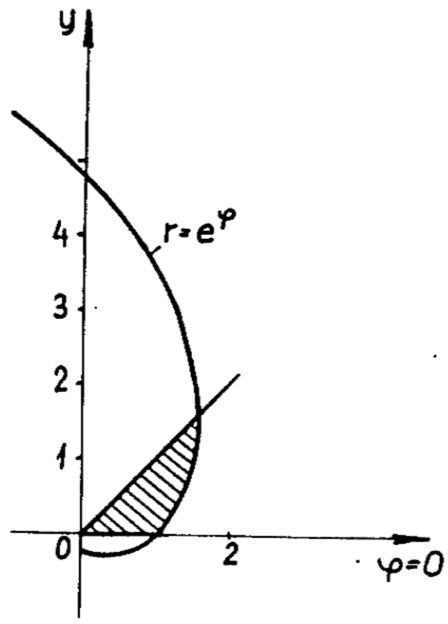

Figure 4: The domain of Exercise 4.

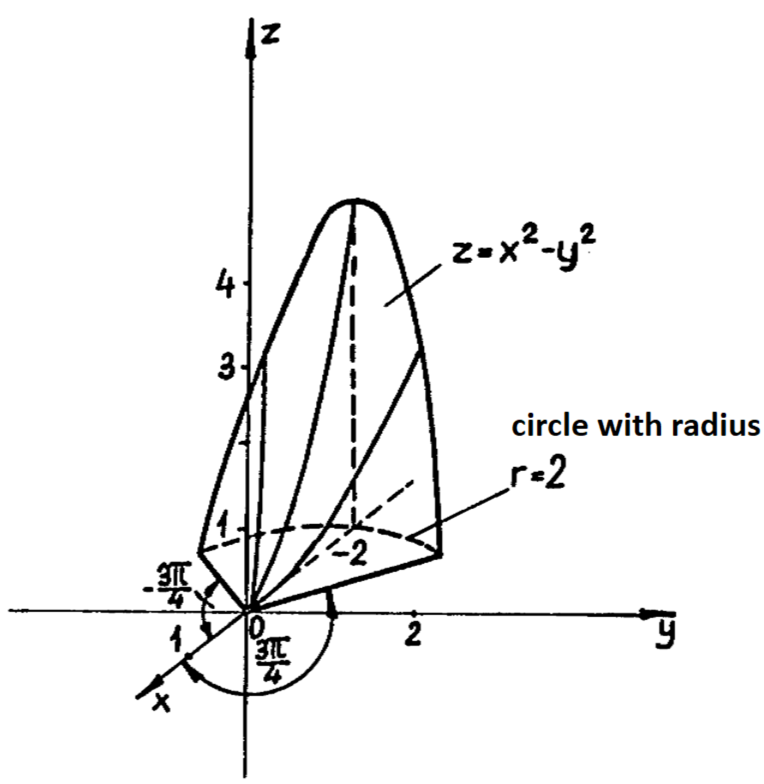

Figure 5: The domain of Exercise 5.

2 Multi-variable integration

6. Calculate the integral of function $f(x,y) = x^2 + y^2$ above the shape given by Figure 6.

   Solution: We can solve this exercise in two different ways: either by using a normal domain with respect to the y-axis, or a normal domain with respect to the x-axis.

   First method (x-axis): In this case $0 \leq x \leq 1$ and $x^2 \leq y \leq \sqrt{x}$, meaning that the integral is

$$\int_0^1 \left(\int_{x^2}^{\sqrt{x}} x^2 + y^2 dy \right) dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} dy \right]_{y=x^2}^{\sqrt{x}} dx = \int_0^1 x^2 \sqrt{x} + \frac{(\sqrt{x})^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} dx =$$

$$= \int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3} dx = \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{3 \cdot 5/2} - \frac{x^5}{5} - \frac{x^7}{3 \cdot 7} \right]_0^1 = \frac{1}{7/2} + \frac{1}{3 \cdot 5/2} - \frac{1}{5} - \frac{1}{3 \cdot 7}$$

   Second method (y-axis): In this case $0 \leq y \leq 1$ and $y^2 \leq x \leq \sqrt{y}$: the reason for these is that from $\sqrt{x} = y$ we get $x = y^2$ (the 'upper' curve) and similarly, from $x^2 = y$ we get $x = \sqrt{y}$ - the reason for the order is that now we move from left to right on the domain so the initial endpoint will be the $x = y^2$ curve and the upper boundary is $x = \sqrt{y}$. Therefore,

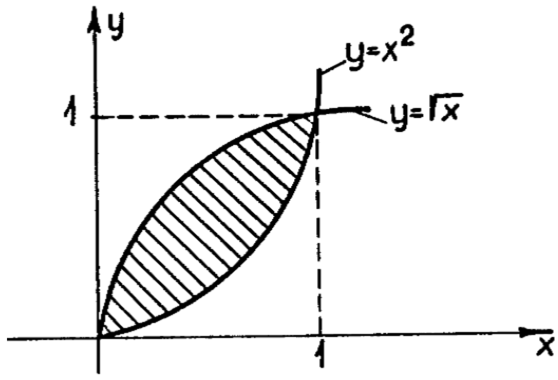

Figure 6: The domain of Exercise 6.

the integral is

$$\int_0^1 \left(\int_{y^2}^{\sqrt{y}} x^2 + y^2 dx \right) dy = \int_0^1 \left[\frac{x^3}{3} + xy^2 dx \right]_{x=y^2}^{\sqrt{y}} dy = \int_0^1 \frac{(\sqrt{y})^3}{3} + y^2\sqrt{y} - \frac{(y^2)^3}{3} - y^2y^2 dy =$$

$$= \left[\frac{y^{5/2}}{3\cdot 5/2} + \frac{y^{7/2}}{7/2} - \frac{y^7}{3\cdot 7} - \frac{y^5}{5} \right]_0^1 = \frac{1}{3\cdot 5/2} + \frac{1}{7/2} - \frac{1}{3\cdot 7} - \frac{1}{5}.$$

7. Change the order of the integrals in the following expression!

$$\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y) dy dx.$$

Solution: The domain is $0 \le x \le 2$, $\sqrt{2x - x^2} \le y \le \sqrt{2x}$, which can be seen on Figure 7.

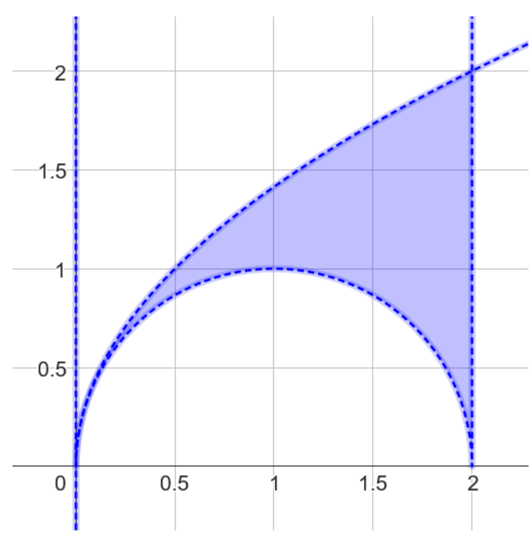

Figure 7: The domain of Exercise 7.

Right now the integral is taken on a normal domain with respect tot the x-axis, and if we would like to change the order of the integrals, we would need to consider it as a normal domain with respect to the y-axis. However, it can be seen that depending on the observed value of y, the horizontal lines might have two disjoint parts if $y < 1$, or just one part when $y > 1$. Because of this, let us split our domain into three parts (see Figure 8) and observe those parts separately.

– <u>*Part 1:*</u> Here $0 \le y \le 1$, and for the two curves: the left one is given by $y = \sqrt{2x}$, meaning that $x = \frac{y^2}{2}$. For the right one, we have $y = \sqrt{2x - x^2}$, and then from this one:

$$y^2 = 2x - x^2,$$
$$x^2 - 2x + y^2 = 0,$$
$$x_{1,2} = \frac{2 \pm \sqrt{(-2)^2 - 4y^2}}{2} = 1 - \sqrt{1 - y^2}.$$

At the last step, we used that the $-\sqrt{(-2)^2 - 4y^2}$ corresponds to that branch which we have to consider (this is the smaller one, meaning that this is the left one). Therefore, $\frac{y^2}{2} \le x \le 1 - \sqrt{1 - y^2}$.

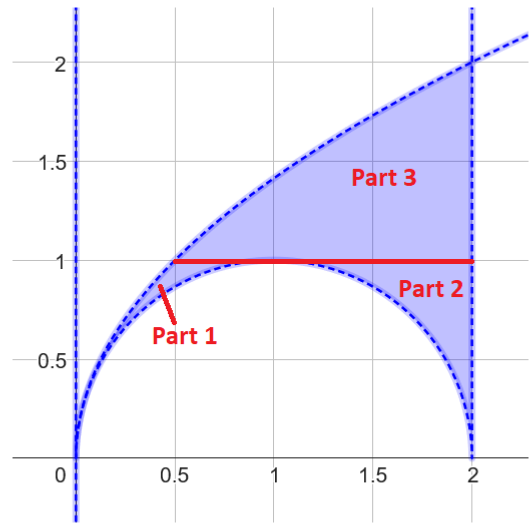

Figure 8: The split of the domain of Exercise 7.

- *Part 2:* Here $0 \leq y \leq 1$, and for the two curves: the right one is just the line $x = 2$, and for the left one we get the other branch of the curve we discussed in the previous case, meaning that we get $y = 1 + \sqrt{1 - y^2}$. Therefore, $\frac{y^2}{2} \leq x \leq 1 + \sqrt{1 - y^2}$.
- *Part 3:* Here $1 \leq y \leq 2$, and for the two curves: the left one is $x = \frac{y^2}{2}$, and the right one is the straight line $x = 2$.

In conclusion, the integral we get is

$$\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x, y) dy dx =$$

$$= \int_0^1 \int_{y^2/2}^{1-\sqrt{1-y^2}} f(x, y) dx dy + \int_0^1 \int_{1+\sqrt{1-y^2}}^{2} f(x, y) dx dy + \int_0^2 \int_{y^2/2}^{2} f(x, y) dx dy.$$

The next exercise shows the practical use of the previous method:

8. Calculate the following integral:

$$\int_0^2 \int_{y/2}^1 y \cos(x^3) dx dy.$$

Solution: If we would like to solve this exercise in its current form, we would have to calculate the indefinite integral $\int \cos(x^3) dx$, which has no closed form (one might get a formula involving gamma functions but we do not discuss those in this course). Because of this, we would have to change the order of the integrals. In the current form, the domain of integration is $\frac{y}{2} \leq x \leq 1$ and $0 \leq y \leq 2$.

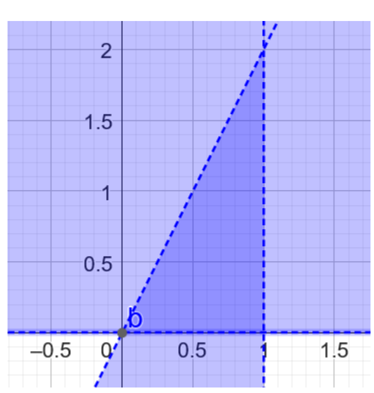

Figure 9: The domain of Exercise 8 is the dark blue shape.

To change the domain, we would move with y from the bottom line $y = 0$ to the upper straight line of $y = 2x$. Then, the domain is given by $0 \leq x \leq 1$ and $0 \leq y \leq 2x$. Therefore, the integral is

$$\int_0^2 \int_{y/2}^1 y \cos(x^3) dx dy = \int_0^1 \int_0^{2x} y \cos(x^3) dy dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{2x} \cos(x^3) dx =$$

$$= \int_0^1 2x^2\cos(x^3)dx = \frac{2}{3}\int_0^1 3x^2\cos(x^3)dx = \frac{2}{3}\left[\sin(x^3)\right]_0^1 = \frac{2}{3}\sin(1).$$

9. Calculate the integral of the function $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$ on the domain T given by the equations $1 \leq x^2 + y^2 \leq 4$ and $x \geq 0$.

Solution: The domain in question can be seen on Figure 10. As we can see, this is a circular domain, so we should use polar coordinates.

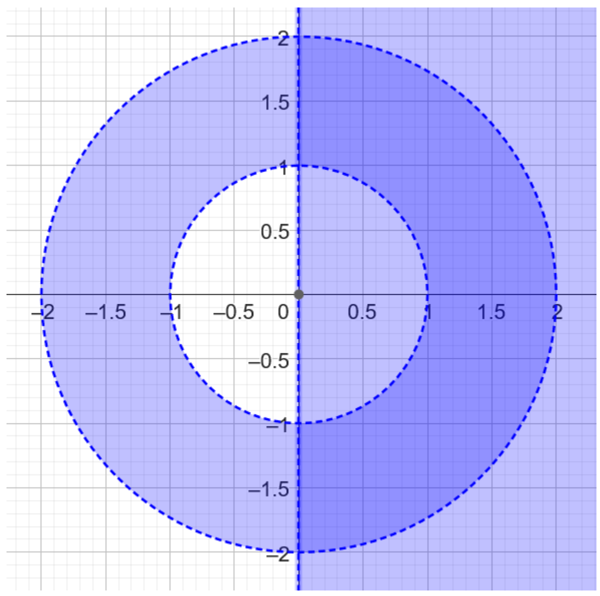

Figure 10: The domain of Exercise 9 is the dark blue shape.

In this case $1 \leq r \leq 2$ and $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. Since $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ and the Jacobian determinant of this transformation is r, we get the following integral:

$$\iint_T \sin\left(\sqrt{x^2+y^2}\right) dT = \int_{-\pi/2}^{\pi/2}\int_1^2 \sin(r)\cdot r\; drd\varphi =$$

Now we solve this integral by integration by parts:

$$\int r\sin(r)dr = -r\cos(r) - \int -\cos(r)dr,$$

and then, by continuing the previous calculation:

$$= \int_{-\pi/2}^{\pi/2}\left(\left[-r\cos(r)\right]_1^2 - \int_1^2 -\cos(r)dr\right) =$$

$$= \int_{-\pi/2}^{\pi/2}\left(\left[-r\cos(r)\right]_1^2 + \left[\sin(r)\right]_1^2\right) d\varphi =$$

$$= \left(\left[-2\cos(2)+\cos(1)\right] + \left[\sin(2)-\sin(1)\right]\right)\int_{-\pi/2}^{\pi/2} 1 d\varphi =$$

$$= \left(\left[-2\cos(2)+\cos(1)\right] + \left[\sin(2)-\sin(1)\right]\right)\left[\varphi\right]_{-\pi/2}^{\pi/2} =$$

$$= \left(\left[-2\cos(2)+\cos(1)\right] + \left[\sin(2)-\sin(1)\right]\right)\pi.$$

10. Transform the integral

$$\iint_T f(x, y)dT$$

into polar coordinates where the domain is given by Figure 11.

Solution: Here $\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$. For the radius, the problem here is that the center of the circle is not at the origin. The way we can solve this problem is to calculate the distance of the curve from the origin for a given value of φ, i.e. calculate the distance of the origin and point P for a given value of φ. Now, if we connect this point P to the point $(1, 0)$, we get a right triangle by Thales's theorem (see Figure 12).

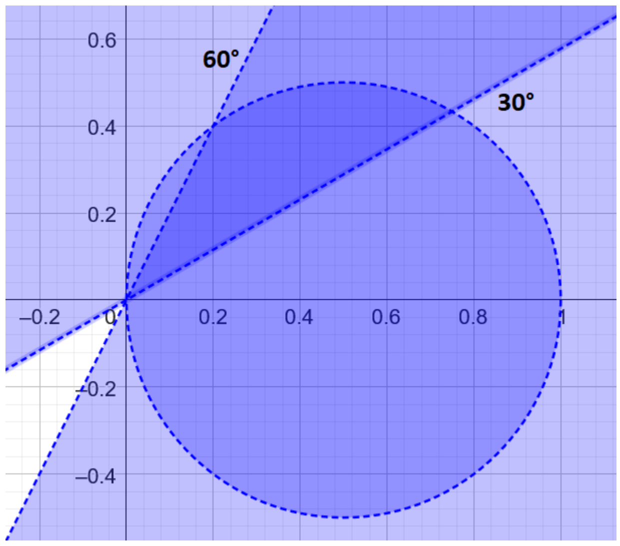

Figure 11: The domain of Exercise 10 is the dark blue shape.

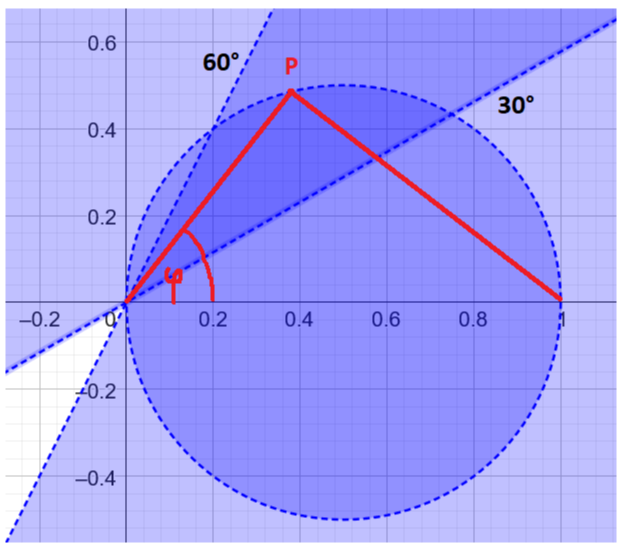

Figure 12: The right triangle inside the domain.

Since this is a right triangle, we can use the definition of the cosine function here: we get that if r_{max} is the distance we are looking for, then

$$\cos(\varphi) = \frac{r_{max}}{1},$$

meaning that $r_{nmax} = \cos(\varphi)$, and then $0 \le r \le \cos(\varphi)$. In conclusion, the integral is

$$\iint_T f(x,y)dT = \int_{\pi/6}^{\pi/3} \int_0^{\cos(\varphi)} r f(r\cos(\varphi), r\sin(\varphi)) dr d\varphi.$$

In this case the order of the integrals is important (the usual rule is that the one with the constants should be outside and the one with the variables should be inside).

11. Transform the integral

$$\iint_T f(x,y)dT$$

into polar coordinates where the domain is given by Figure 13.

Solution: It is clear that $-\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$. Moreover, to calculate the distance of the large circle from the origin, we can use the Thales's theorem like in the previous exercise, from which we get that $r_{max} = 2\cos(\varphi)$, so $\frac{1}{2} \le r \le 2\cos(\varphi)$, and then

$$\iint_T f(x,y)dT = \int_{-\pi/4}^{\pi/4} \int_{1/2}^{2\cos(\varphi)} r f(r\cos(\varphi), r\sin(\varphi)) dr d\varphi.$$

12. Calculate the volume of the space which is below the graph of the function $f(x,y) = x^2 + 2y^2$ and the section of the $x - y$ plane given by Figure 14.

Solution: The question of this exercise is the integral of function f above the domain given by the figure. Then, since we have a circular domain, we should use polar coordinates: the

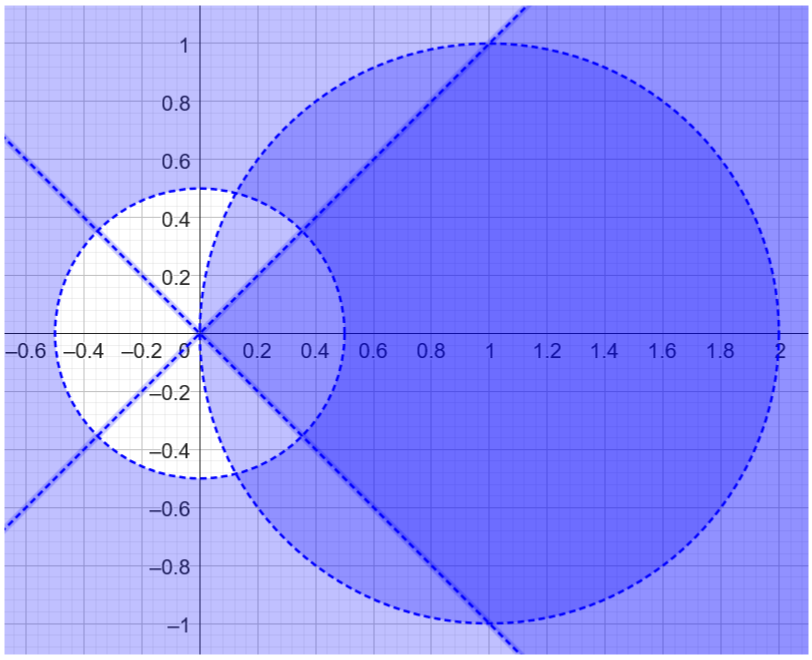

Figure 13: The domain of Exercise 11 is the dark blue region.

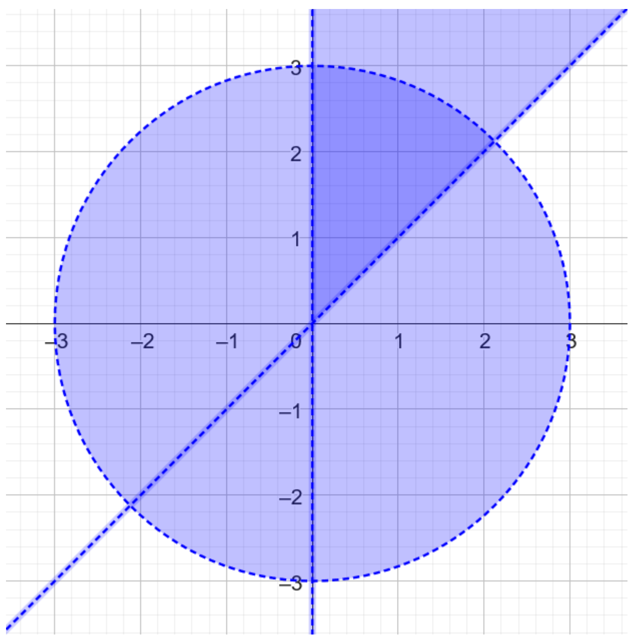

Figure 14: The domain of Exercise 12 is the dark blue region.

domain can be described by $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$ and $0 \leq r \leq 3$. Then, the integral is (by using the rules $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ and that the Jacobian determinant is r):

$$\iint_t f(x,y)dxdy = \int_{\pi/4}^{\pi/2}\int_0^3 \left(r^2\cos^2(\varphi) + 2r^2\sin^2(\varphi)\right) r\ drd\varphi =$$

$$= \int_{\pi/4}^{\pi/2}\left(\int_0^3 r^3\cos^2(\varphi) + 2r^3\sin^2(\varphi)dr\right)d\varphi = \int_{\pi/4}^{\pi/2}\left(\cos^2(\varphi)\int_0^3 r^3dr + 2\sin^2(\varphi)\int_0^3 r^3dr\right)d\varphi =$$

$$= \int_{\pi/4}^{\pi/2}\left(\cos^2(\varphi)\left[\frac{r^4}{4}\right]_0^3 + 2\sin^2(\varphi)\left[\frac{r^4}{4}\right]_0^3\right)d\varphi = \int_{\pi/4}^{\pi/2}\left(\cos^2(\varphi)\frac{3^4}{4} + 2\sin^2(\varphi)\frac{3^4}{4}\right)d\varphi =$$

$$= \frac{81}{4}\int_{\pi/4}^{\pi/2}\cos^2(\varphi) + 2\sin^2(\varphi)d\varphi = \frac{81}{4}\int_{\pi/4}^{\pi/2} 1 + \sin^2(\varphi)d\varphi =$$

Now we can use the fact that $\sin^2(\varphi) = \frac{1-\cos(2\varphi)}{2}$, meaning that

$$= \frac{81}{4}\int_{\pi/4}^{\pi/2} 1 + \frac{1-\cos(2\varphi)}{2}d\varphi = \frac{81}{4}\int_{\pi/4}^{\pi/2}\frac{3}{2} - \frac{\cos(2\varphi)}{2}d\varphi = \frac{81}{4}\left[\frac{3}{2}\varphi - \frac{\sin(2\varphi)}{4}\right]_{\pi/4}^{\pi/2} =$$

$$= \frac{81}{4}\left[\frac{3}{2}\frac{\pi}{2} - \left(\frac{3}{2}\frac{\pi}{4} - \frac{1}{4}\right)\right] = \frac{81}{4}\left[\frac{3}{8}\pi + \frac{1}{4}\right]$$