Math G2 Practice 2

Matrices: inverse, rank.

An **inverse** of an n-by-n matrix A is such a matrix A^{-1} for which we have $A \cdot A^{-1} = A^{-1} \cdot A = I$, where I is the n-by-n identity matrix (so a matrix with 1s in its main diagonal and 0s everywhere else).

The inverse of a matrix can be done in different ways:

1. By elementary methods (not recommended, see the Lecture)
2. By using the Gaussian elimination (see Practice 3).
3. By using the **adjoint** of the matrix. This method uses the following statement:

   $$A^{-1} = \frac{1}{\det(A)} \cdot A^*,$$

   where A^* is the adjoint of matrix A. Here we can see that this formula only works if $\det(A) \neq 0$, but we do not really have to worry about this fact, since:

   Theorem: For an n-by-n matrix A, the following two are equivalent:

   – $\det(A) \neq 0$,
   – A^{-1} exists.

   Now the only remaining question is: how to compute the adjoint of matrix A. This has three steps:

   – First we construct a matrix of minors,
   – then we change the signs of this matrix using the chessboard rule, and
   – take the transpose of this matrix.

In this session we are going to use the third method involving the adjoint matrix.

1. Has the following matrix an inverse? If yes, calculate it using its adjoint.

   $$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -2 \\ -5 & -4 & -1 \end{pmatrix}$$

   Solution: For the first question, we are going to use the determinant of this matrix:

   $$\det(A) = 1 \cdot \begin{vmatrix} 3 & -2 \\ -4 & -1 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 4 & -2 \\ -5 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & 3 \\ -5 & -4 \end{vmatrix} =$$
   $$= (-3 - 8) - 2(-4 - 10) + (-16 + 15) = 16 \neq 0,$$

   so the matrix has an inverse.

   Now we start to construct the adjoint of matrix A.

   1. The first step is to calculate the matrix of minors: this is a matrix containing the minors of each element as entries. (Minor of an element: the determinant of the smaller matrix which we get in a way if we delete the row and column of this given element.) So it means that at position (1,1) we will have the minor of element $a_{1,1}$, at position (1,2) we have the minor of element $a_{1,2}$ and so on. So the matrix of minors have the following form:

      $$\begin{pmatrix} \begin{vmatrix} 3 & -2 \\ -4 & -1 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ -5 & -1 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ -5 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -4 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -5 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -5 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} -11 & -14 & -1 \\ 2 & 4 & 6 \\ -7 & -6 & -5 \end{pmatrix}$$

2. The next step is to change the signs according to the chessboard rule:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

   We change those signs where we have a $-$ and leave those which have a $+$. So it means that our matrix will have the form

$$\begin{pmatrix} -11 & 14 & -1 \\ -2 & 4 & -6 \\ -7 & 6 & -5 \end{pmatrix}.$$

3. The last step is to take the transpose of this matrix:

$$\begin{pmatrix} -11 & -2 & -7 \\ 14 & 4 & 6 \\ -1 & -6 & -5 \end{pmatrix}.$$

So this is the adjoint of our matrix, which means that the inverse of matrix A is:

$$A^{-1} = \frac{1}{16} \begin{pmatrix} -11 & -2 & -7 \\ 14 & 4 & 6 \\ -1 & -6 & -5 \end{pmatrix} = \begin{pmatrix} -11/16 & -1/8 & -7/16 \\ 7/8 & 1/4 & 3/8 \\ -1/16 & -3/8 & -5/16 \end{pmatrix}.$$

It can be shown that in this case $A \cdot A^{-1} = A^{-1} \cdot A = I$ holds.

2. Has the following matrix an inverse? If yes, calculate it using its adjoint.

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix}$$

Solution: For the first question, we are going to use the determinant of this matrix:

$$\det(A) = 2 \cdot \begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 3 & -2 \\ 3 & 4 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 3 & 4 \\ 3 & -2 \end{vmatrix} =$$

$$= 2(16 - 4) + 2(12 - (-6)) - (-6 - 12) = 78 \neq 0$$

so the matrix has an inverse.

The first step is to calculate the matrix of minors: it has the form

$$\begin{pmatrix} \begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} -1 & -1 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} -1 & -1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 12 & 18 & -18 \\ -6 & 11 & -1 \\ 6 & 1 & 11 \end{pmatrix}$$

Now we change the sign according to the chessboard rule:

$$\begin{pmatrix} 12 & -18 & -18 \\ 6 & 11 & 1 \\ 6 & -1 & 11 \end{pmatrix}$$

Then, we take the transpose of this previous matrix:

$$\begin{pmatrix} 12 & 6 & 6 \\ -18 & 11 & -1 \\ -18 & 1 & 11 \end{pmatrix}$$

So the inverse of matrix A is:

$$A^{-1} = \frac{1}{78}\begin{pmatrix} 12 & 6 & 6 \\ -18 & 11 & -1 \\ -18 & 1 & 11 \end{pmatrix} = \begin{pmatrix} 2/13 & 1/13 & 1/13 \\ -3/13 & 11/78 & -1/78 \\ -3/13 & 1/78 & 11/78 \end{pmatrix}$$

3. Let us assume that we know that for some 2-by-2 matrix X the following holds:

$$X \cdot \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix}.$$

What is X?

Solution: The matrix-equation we have to solve is

$$X \cdot A = B.$$

Let us multiply both sides by A^{-1}! Then,

$$X \cdot A \cdot A^{-1} = B \cdot A^{-1},$$

but since $A \cdot A^{-1} = I$ and $X \cdot I = X$, the equation we have is

$$X = B \cdot A^{-1},$$

so we have to calculate the inverse of A. Since $\det(A) = -1 \neq 0$, it exists.

Fortunately, A is only a 2-by-2 matrix, so its inverse can be calculated pretty easily: the minors are basically just the elements we have after the deletion of the rows and columns:

$$\begin{pmatrix} 5 & 4 \\ 9 & 7 \end{pmatrix}$$

Now we change the signs:

$$\begin{pmatrix} 5 & -4 \\ -9 & 7 \end{pmatrix},$$

and then we take the transpose:

$$\begin{pmatrix} 5 & -9 \\ -4 & 7 \end{pmatrix}.$$

Then, the inverse of A is:

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -9 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}.$$

So the final solution is

$$X = B \cdot A^{-1} = \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix} = \begin{pmatrix} -16 & 29 \\ 17 & -30 \end{pmatrix}.$$

The **rank** of a matrix is the number of linearly independent rows (or columns) of the matrix. Since it is not changed during the following transformations:

- addition of a row (possibly multiplied by a real number) to another one,
- multiplying one row by a (nonzero) number,
- changing two rows or columns,

our main goal will always be to transform our matrix into an upper-triangular form.

Proposition: For an upper triangular matrix, its rank is the number of its non-zero (i.e. not all-zero) rows (or columns, if their number is smaller).

So in the end of these exercises we will always count the non-zero rows, and this will be the rank of our matrix.

Useful fact: If two (non-zero) vectors have zeros at different positions, then they are independent. It means that e.g. if we have a vector v_1 which has a zero at position 5 but another vector v_2 does not, then they are linearly independent.

4. What is the rank of the following matrix?

$$\begin{pmatrix} 3 & 4 & 1 & 1 \\ -4 & -3 & -6 & -1 \\ 1 & 1 & 1 & -1 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

Solution: To make our calculations easier, we are going to change the order of the rows: we will move row 3 into the position of row 1 (it is easier to use 1 than 3). Then, our matrix is

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ -4 & -3 & -6 & -1 \\ 3 & 4 & 1 & 1 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

Now we move forward as in the exercises in the previous practice session: we multiply row 1 by 4 and add it to the second one:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -5 \\ 3 & 4 & 1 & 1 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

Then, we multiply row 1 by 3 and subtract it from row 3:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 1 & -2 & 4 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

Then, we multiply row 1 by 2 and subtract it from the last row:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For element $a_{3,2} = 1$, we use row 2: we simply subtract it form row 3:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now we have to realize that row 3 and 4 are dependent, since when we multiply row 3 by 1/9 we get row 4: so the rank can be at most 3. In other words, if we multiply row 3 by 1/9 and subtract from row 4, we get

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the rank can be at most 3. Is it 3 though?

The rank is 3, if the rows $(1,1,1,-1)$, $(0,1,-2,4)$ and $(0,0,0,1)$ are linearly independent: this means that the equation

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

can only hold if $c_1 = c_2 = c_3 = 0$. For the first coordinate we have $c_1 = 0$, then for the second one $c_1 + c_2 = 0$, but since $c_1 = 0$ then $c_2 = 0$. Also, from the last coordinate we have

$$-c_1 + 4c_2 + 9c_3 = 0,$$

but since $c_1 = c_2 = 0$, this can only hold if $c_3 = 0$. Therefore, these three rows are independent, so the rank is 3.

Remark: It is also possible to continue the elimination process until we have a diagonal matrix: let us divide row 3 by 9 and change columns 3 and 4:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us subtract column 1 from columns 2 and 4, and add it to column 3:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, multiply column 2 by 5 and add it to column 3, and similarly multiply column 2 by 2 and add it to column 4:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, we have a diagonal matrix, and the rank of a diagonal matrix is always the number of non-zero elements in its main diagonal.

5. What is the rank of the following matrix?

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix}$$

Solution: In Exercise 2. we saw that this matrix has an inverse. Then, we can use the following theorem:

Theorem: The following properties are equivalent for an n-by-n matrix:

- $\det(A) \neq 0$,
- A^{-1} exists,
- $\text{rank}(A) = n$.

Then, since we know that an inverse exists, the rank of this matrix is maximal, i.e. $\text{rank}(A) = 3$.

6. What is the rank of the following matrix?

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 9 & 3 \\ 1 & 3 & 1 & 8 \\ 1 & 1 & 4 & 2 \end{pmatrix}$$

Solution: Since the column-rank and the row-rank is the same and the matrix has four columns, its rank is at most 4. Let us start our process by a row change: let us change rows 1 and 2 to make our computations easier.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & 1 \\ 1 & 2 & 9 & 3 \\ 1 & 3 & 1 & 8 \\ 1 & 1 & 4 & 2 \end{pmatrix}.$$

Then, by multiplying row 1 by 3 (for row 2) and simply leaving it as it is for the others, after subtracting it we get

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & 1 & 7 & 2 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

To make the next steps easier, let us change rows 2 and 3:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & -2 & -5 & -2 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

Then, by multiplying row 2 by 2 and adding it to row 3 and subtracting it from row 4, we have

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 9 & 2 \\ 0 & 0 & -15 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

To make the following steps easier, let us change row 3 and 5:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -15 & 3 \\ 0 & 0 & 9 & 2 \end{pmatrix}.$$

Let us also divide row 4 by -3:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 9 & 2 \end{pmatrix}.$$

Now let us multiply rows 4 and 5 by 2:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 10 & -2 \\ 0 & 0 & 18 & 4 \end{pmatrix}.$$

Then, by multiplying row 3 by 5 for row 4 and by 9 for row 5 and then subtracting it from the proper rows, we get the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -5 \end{pmatrix}.$$

Now, if we multiply row 4 by 5/7 and add it to the last one, we have

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the rank can be at least 4. However, by the 'Useful fact' mentioned at the beginning of this section, all the columns are now linearly independent, so the rank is indeed 4. (Alternatively, we can continue the process and get a diagonal matrix with four 1s in its main diagonal.)