Math G2 Practice 3

Systems of Linear Algebraic Equations: the Gaussian elimination

For the solution of systems of linear algebraic equations (SLAE in short) we will use the method known as Gaussian elimination. During this process the main idea is that if we have a SLAE in the form

$$Ax = b,$$

where A is the matrix of coefficients, x is the vector of unknowns and b is a vector of known values (the 'right-hand side'), then we construct the extended matrix $\hat{A} = (A|b)$ (which has the $n \times n$ matrix A as its first n-many columns and b is its last column) and we try to transform it into an upper triangular form by using the following three operations:

1. add one of the rows (possibly multiplied by some real number) to another row,
2. multiply one of the rows,
3. change the order of two rows.

Then, in the end we get an upper triangular matrix, in which case we can proceed further in two different ways:

- we can either solve this equation in this form (this is easy, see the lecture for details),
- or continue the process until we have an identity matrix on the left of our matrix: then, the solution is in the last column of the matrix.

An important statement is the following theorem:

Theorem 1. *The following statements are equivalent:*

- *The system $Ax = b$ has a unique solution.*
- *A is invertible.*
- $\det(A) \neq 0$*,*
- $\operatorname{rank}(A) = n$ *(here n is the number of unknowns).*

Another theorem is about the number of solutions:

Theorem 2. *Let us consider the extended matrix $\hat{A} = (A|b)$ and let us denote the number of solutions by n. Then,*

- *if* $\operatorname{rank}(A) = \operatorname{rank}(\hat{A}) = n$*, then the system $Ax = b$ has a unique solution,*
- *if* $\operatorname{rank}(A) < \operatorname{rank}(\hat{A})$*, then the system $Ax = b$ has no solution,*
- *if* $\operatorname{rank}(A) = \operatorname{rank}(\hat{A}) < n$*, then the system $Ax = b$ has infinitely many solutions.*

Exercises

1. Does the following system of equations have a unique solution? If yes, solve it!

$$\begin{cases} 2x_1 - x_2 - x_3 = 4, \\ 3x_1 + 4x_2 - 2x_3 = 11, \\ 3x_1 - 2x_2 + 4x_3 = 11. \end{cases}$$

Solution: This SLAE has the following matrix form:

$$\begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}$$

The easiest way to determine whether this SLAE has a solution or not is to calculate the determinant of A - however, in this case we saw it in Practice 2 that A has an inverse (Exercise 2), which was

$$A^{-1} = \frac{1}{60}\begin{pmatrix} 12 & 6 & 6 \\ -18 & 11 & 1 \\ -18 & 1 & 11 \end{pmatrix}.$$

If someone knows the inverse of A, then the solution can easily be calculated by the following argument:

$$Ax = b \qquad \Leftrightarrow \qquad A^{-1}Ax = A^{-1}b \qquad \Leftrightarrow \qquad x = A^{-1}b,$$

meaning that

$$x = A^{-1}b = \frac{1}{60}\begin{pmatrix} 12 & 6 & 6 \\ -18 & 11 & 1 \\ -18 & 1 & 11 \end{pmatrix}\begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix},$$

meaning that the solution is $x_1 = 3$, $x_2 = 1$ and $x_3 = 1$.

Remark: Because of this, in the midterm you will have two methods to solve an SLAE: you either use Gaussian elimination, or you calculate the inverse of matrix A (by the method discussed in Practice 2) and use the formula $x = A^{-1}b$.

2. Solve the following SLAE! How many solutions does it have?

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 - x_5 = 0, \\ 2x_1 + x_2 - x_3 - x_4 + x_5 = 0, \\ x_1 + 7x_2 - 5x_3 - 5x_4 + 5x_5 = 0, \\ 3x_1 - x_2 - 2x_3 + x_4 + x_5 = 0. \end{cases}$$

Solution: Since here we have only four equations and five unknowns, our assumption is that this system might have infinitely many solutions. The matrix form in this case is

$$\begin{pmatrix} 1 & -2 & 1 & 1 & -1 \\ 2 & 1 & -1 & -1 & 1 \\ 1 & 7 & -5 & -5 & 5 \\ 3 & -1 & -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since A is not a square matrix, we cannot calculate its inverse: because of this, we will use Gaussian elimination. In this case the extended matrix is

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & -1 & 1 & 0 \\ 1 & 7 & -5 & -5 & 5 & 0 \\ 3 & -1 & -2 & 1 & 1 & 0 \end{array}\right).$$

Our goal is to transform it into an upper triangular form, i.e. make the red elements disappear. Let us multiply the first row by 2 and subtract it from the second row:

$$\begin{array}{r} (\; 2 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\; 0 \;) \\ -(\; 2 \;\; -4 \;\; 2 \;\; 2 \;\; -2 \;\; 0 \;) \\ \hline (\; 0 \;\; 5 \;\; -3 \;\; -3 \;\; 3 \;\; 0 \;) \end{array}$$

Then, the matrix has the form

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 5 & -3 & -3 & 3 & 0 \\ 1 & 7 & -5 & -5 & 5 & 0 \\ 3 & -1 & -2 & 1 & 1 & 0 \end{array}\right).$$

Now let us subtract the first row from the second row:

$$\begin{array}{r} (\; 1 \;\; 7 \;\; -5 \;\; -5 \;\; 5 \;\; 0 \;) \\ -(\; 1 \;\; -2 \;\; 1 \;\; 1 \;\; -1 \;\; 0 \;) \\ \hline (\; 0 \;\; 9 \;\; -6 \;\; -6 \;\; 6 \;\; 0 \;) \end{array}$$

Then, the matrix has the form

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 5 & -3 & -3 & 3 & 0 \\ 0 & 9 & -6 & -6 & 6 & 0 \\ 3 & -1 & -2 & 1 & 1 & 0 \end{array}\right).$$

Now let us multiply the first row by 3 and subtract it from the last one:

$$\begin{array}{r} \begin{pmatrix} 3 & -1 & -2 & 1 & 1 & 0 \end{pmatrix} \\ -\begin{pmatrix} 3 & -6 & 3 & 3 & -3 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 5 & -5 & -2 & 4 & 0 \end{pmatrix} \end{array}$$

Then, the matrix has the form

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 5 & -3 & -3 & 3 & 0 \\ 0 & 9 & -6 & -6 & 6 & 0 \\ 0 & 5 & -5 & -2 & 4 & 0 \end{array}\right).$$

Now let us subtract row 2 from row 4:

$$\begin{array}{r} \begin{pmatrix} 0 & 5 & -5 & -2 & 4 & 0 \end{pmatrix} \\ -\begin{pmatrix} 0 & 5 & -3 & -3 & 3 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & -2 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Then, the matrix has the form

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 5 & -3 & -3 & 3 & 0 \\ 0 & 9 & -6 & -6 & 6 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right).$$

To make our calculations easier, let us divide row 3 by 3:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 5 & -3 & -3 & 3 & 0 \\ 0 & 3 & -2 & -2 & 2 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right).$$

Then, multiply row 3 by 5 and row 2 by 3:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 15 & -9 & -9 & 9 & 0 \\ 0 & 15 & -10 & -10 & 10 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right).$$

Now subtract row 2 to row 3:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 15 & -9 & -9 & 9 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right).$$

Now multiply row 3 by 2 and subtract it from row 4:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 0 \\ 0 & 15 & -9 & -9 & 9 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 0 \end{array}\right).$$

Now we can see that here the rank of the matrix is 4, but we have 5 unknowns, meaning that we have infinitely many solutions. Also, the degree of freedom here is $n-\text{rank}(A) = 5-4 = 1$, meaning that we will have one free variable - let us choose x_5 to be this one, so our one parameter is $x_5 = p$. Then, the current form of equation 4 is

$$3x_4 - x_5 = 0,$$

from which we get that $x_4 = \dfrac{x_5}{3} = \dfrac{p}{3}$. Similarly, equation 3 is

$$-x_3 - x_4 + x_5 = 0,$$

from which we get $x_3 = \dfrac{2}{3}p$. By the same methods, we can get $x_2 = 0$ and $x_1 = 0$. Then, the final solution is $x = \begin{pmatrix} 0 \\ 0 \\ 2/3p \\ 1/3p \\ p \end{pmatrix}$.

Remark: In this case one can also choose x_4 (or even x_3) to be their free variable: we get a solution which is equivalent to the one above.

3. Solve the following SLAE! How many solutions does it have?

$$\begin{cases} x_1 + \ \ x_2 - 3x_3 = -1, \\ 2x_1 + \ \ x_2 - 2x_3 = \ \ 1, \\ x_1 + \ \ x_2 + \ \ x_3 = \ \ 3, \\ x_1 + 2x_2 - 3x_3 = \ \ 1. \end{cases}$$

Solution: The matrix form in this case is

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

Since we have 3 unknown and 4 equations, we might suspect that it has no solution. The extended matrix has the form

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & 1 \end{array}\right).$$

Let us multiply the first row by 2 and subtract it from the second one:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & 1 \end{array}\right).$$

Now let us subtract the first row from the third and the fourth rows:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 2 \end{array}\right).$$

Let us add row 2 to the last row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 5 \end{array}\right).$$

Let us subtract row 3 from the last row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This means that the rank of A is 3, but the rank of the extended matrix is 4: because of this, it will have no solution.

Another way to see this is to write up the current form of the equations:

$$\begin{cases} x_1 + x_2 - 3x_3 = -\,1, \\ \quad -x_2 + 4x_3 = \ 3, \\ \qquad\quad 4x_3 = \ 4, \\ \qquad\qquad 0 = \ 1. \end{cases}$$

As we can see, the last equation is clearly a contradiction.

4. Solve the following SLAE! How many solutions does it have depending on the value of λ?

$$\begin{cases} \lambda x_1 + x_2 + x_3 + x_4 = 1, \\ x_1 + (1+\lambda)x_2 + x_3 + x_4 = 3, \\ x_1 + x_2 + (1+\lambda)x_3 + x_4 = 4, \\ x_1 + x_2 + x_3 + x_4 = 1. \end{cases}$$

Solution: The matrix form in this case is

$$\begin{pmatrix} \lambda & 1 & 1 & 1 \\ 1 & 1+\lambda & 1 & 1 \\ 1 & 1 & 1+\lambda & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 1 \end{pmatrix}$$

Here we have 4 equations with 4 unknowns, so we might have a unique solution. The extended matrix is

$$\left(\begin{array}{cccc|c} \lambda & 1 & 1 & 1 & 1 \\ 1 & 1+\lambda & 1 & 1 & 3 \\ 1 & 1 & 1+\lambda & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

To simplify the process, let us change the order of rows 4 and 1:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1+\lambda & 1 & 1 & 3 \\ 1 & 1 & 1+\lambda & 1 & 4 \\ \lambda & 1 & 1 & 1 & 1 \end{array}\right).$$

If we continue the process in this way, we might have to multiply one of the rows by $1/\lambda$, but we do not know whether λ is zero or not: because of this, the trick we are going to apply is to change the columns of 1 and 4:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1+\lambda & 1 & 1 & 3 \\ 1 & 1 & 1+\lambda & 1 & 4 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right).$$

Keep in mind though that in this case column 1 corresponds to x_4 and column 4 has the coefficients of x_1.

Now let us subtract the first row from all the others:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 0 & 0 & 2 \\ 0 & 0 & \lambda & 0 & 3 \\ 0 & 0 & 0 & \lambda-1 & 0 \end{array}\right).$$

Now we have three cases:

(a) *Case 1:* If $\lambda \neq 0$ and $\lambda - 1 \neq 0$ (meaning that $\lambda \neq 1$), then the rank of A and the rank of the extended matrix is the same, namely four, which was the number of unknowns: because of this, in this case we have a unique solution. Our equations in this case are

$$\begin{cases} x_4 + x_2 + x_3 + x_1 = 1, \\ \lambda x_2 = 2, \\ \lambda x_3 = 3, \\ (\lambda - 1)x_1 = 0. \end{cases}$$

(Keep in mind that the first column was describing x_4 and the last one was describing x_1). Then, $x_1 = 0$, $x_3 = \dfrac{3}{\lambda}$, $x_2 = \dfrac{2}{\lambda}$ and $x_4 = 1 - \dfrac{5}{\lambda}$.

(b) <u>*Case 2:*</u> If $\lambda = 1$, then the matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since we have an all-zero row, then the rank of A and the extended matrix is also three, which is smaller than the number of variables, meaning that we have infinitely many solutions. Our equations are in the form

$$\begin{cases} x_4 + x_2 + x_3 + x_1 = & 1, \\ x_2 = & 2, \\ x_3 = & 3, \\ 0 = & 0. \end{cases}$$

Since $4 - 3 = 1$, we have one free variable: let us choose x_1 to be this one, so this will be the parameter $x_1 = p$. Then, $x_2 = 2$, $x_3 = 3$ and $x_4 = -4 - p$.

(c) <u>*Case 3:*</u> If $\lambda = 0$, then the matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right).$$

Here the rank of A is 2 but the rank of the extended matrix is 3, so it has no solution. (Or, another reason is that the final equation has the form $-1 = 0$.)

5. Solve the following SLAE! How many solutions does it have depending on the value of λ?

$$\begin{cases} \lambda x_1 + 2x_2 + 3x_3 + 3x_4 = & 1, \\ 2x_1 + (3+\lambda)x_2 + 6x_3 + 6x_4 = & 1+\lambda, \\ 3x_1 + 6x_2 + (8+\lambda)x_3 + 9x_4 = & 2+\lambda, \\ 3x_1 + 6x_2 + 9x_3 + (8+\lambda)x_4 = & 2+\lambda. \end{cases}$$

Solution: The matrix form in this case is

$$\begin{pmatrix} \lambda & 2 & 3 & 3 \\ 2 & 3+\lambda & 6 & 6 \\ 3 & 6 & 8+\lambda & 9 \\ 3 & 6 & 9 & 8+\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+\lambda \\ 2+\lambda \\ 2+\lambda \end{pmatrix}$$

Here we have 4 equations with 4 unknowns, so we might have a unique solution. The extended matrix is

$$\left(\begin{array}{cccc|c} \lambda & 2 & 3 & 3 & 1 \\ 2 & 3+\lambda & 6 & 6 & 1+\lambda \\ 3 & 6 & 8+\lambda & 9 & 2+\lambda \\ 3 & 6 & 9 & 8+\lambda & 2+\lambda \end{array}\right).$$

To simplify the process, let us change the order of columns 1 and 4:

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & \lambda & 1 \\ 6 & 3+\lambda & 6 & 2 & 1+\lambda \\ 9 & 6 & 8+\lambda & 3 & 2+\lambda \\ 8+\lambda & 6 & 9 & 3 & 2+\lambda \end{array}\right).$$

Let us subtract the first row multiplied by 2 from the second row:

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 2-2\lambda & \lambda-1 \\ 9 & 6 & 8+\lambda & 3 & 2+\lambda \\ 8+\lambda & 6 & 9 & 3 & 2+\lambda \end{array}\right).$$

Now subtract the first row multiplied by 3 from the third row:

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 2-2\lambda & \lambda-1 \\ 0 & 0 & \lambda-1 & 3-3\lambda & \lambda-1 \\ 8+\lambda & 6 & 9 & 3 & 2+\lambda \end{array}\right).$$

Then, subtract the first row multiplied by 3 from the fourth one:

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & \lambda & 1 \\ 0 & \lambda-1 & 0 & 2-2\lambda & \lambda-1 \\ 0 & 0 & \lambda-1 & 3-3\lambda & \lambda-1 \\ \lambda-1 & 0 & 0 & 3-3\lambda & \lambda-1 \end{array}\right).$$

Now we have two cases:

(a) <u>*Case 1:*</u> If $\lambda = 1$ then the matrix is

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Here the rank of A and the extended matrix is both 1, but it is smaller than 4, so we have infinitely many solutions. Moreover, since $4-1=3$, we have three free variables: let us choose these to be $x_2 = p_1$, $x_3 = p_2$ and $x_4 = p_3$, and then $x_1 = 1-3p_3-2p_1-3p_2$ (keep in mind that the first and the last columns were changed).

(b) <u>*Case 2:*</u> If $\lambda \neq 1$,. then let us multiply rows 2, 3 and 4 by $1/(\lambda-1)$: the matrix we get is

$$\left(\begin{array}{cccc|c} 3 & 2 & 3 & \lambda & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & -3 & 1 \end{array}\right).$$

Let us change rows 1 and 4 to make our calculations easier:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 3 & 2 & 3 & \lambda & 1 \end{array}\right).$$

Subtract row 1 multiplied by 3 from row 4:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 2 & 3 & \lambda+9 & -2 \end{array}\right).$$

Subtract row 2 multiplied by 2 from row 4:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 3 & \lambda+13 & -4 \end{array}\right).$$

Subtract row 3 multiplied by 3 from row 4:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & \lambda+22 & -7 \end{array}\right).$$

Now if $\lambda = -22$, we have no solution (since we have $0 = -7$ as our last equation), but if $\lambda \neq -22$, then there is a unique solution (since both of the ranks are 4), and we have $x_1 = \dfrac{-7}{\lambda+22}$, $x_2 = 1 - \dfrac{14}{\lambda+22}$, $x_3 = 1 - \dfrac{21}{\lambda+22}$, $x_4 = 1 - \dfrac{21}{\lambda+22}$ (keep in mind that the first and the last columns were changed).