Math G2 Practice 4

Eigenvalues and eigenvectors

The (real or complex) number λ is an **eigenvalue** of matrix A, if for a non-zero vector v we have

$$Av = \lambda v.$$

Here v is the **eigenvector** corresponding to the eigenvalue λ.

Calculation: The solutions of the **characteristic equation**

$$\det(A - \lambda I) = 0$$

are the eigenvalues of A. Also, the solutions of the equation (for a given λ)

$$(A - \lambda I)v = 0$$

are the eigenvectors corresponding to the eigenvalue λ. For every eigenvalue we have infinitely many eigenvectors, but in most cases (when the eigenvalue has geometric multiplicity one, see the lecture) they are linearly dependent.

1. Calculate the eigenvalues and eigenvectors of the following matrix!

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Solution: Here

$$A - \lambda I = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix}$$

The characteristic equation of matrix A is

$$\det(A - \lambda I) = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8 = 0.$$

The solutions are $\lambda_1 = 4$ and $\lambda_2 = -2$, meaning that these are the eigenvalues of matrix A.

Now let us calculate the eigenvectors.

- The eigenvector of $\lambda_1 = 4$: we have to solve the equation $(A - \lambda I)v = (A - 4I)v = 0$, i.e.

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  These equations are always dependent, meaning that it is enough to just consider one of them, e.g. the first one:

$$-3v_1 + 3v_2 = 0.$$

  Let us choose e.g. $v_1 = p$, then $v_2 = p$, so any vector in the form $\begin{pmatrix} p \\ p \end{pmatrix}$ ($p \in \mathbb{R}$) is an eigenvector of λ_1 - e.g. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector too.

- The eigenvector of $\lambda_2 = -2$: we have to solve the equation $(A - \lambda I)v = (A + 2I)v = 0$, i.e.

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  These equations are always dependent, meaning that it is enough to just consider one of them, e.g. the first one:

$$3v_1 + 3v_2 = 0.$$

  Let us choose e.g. $v_1 = p$, then $v_2 = -p$, so any vector in the form $\begin{pmatrix} p \\ -p \end{pmatrix}$ ($p \in \mathbb{R}$) is an eigenvector of λ_1 - e.g. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector too.

Remark: The second eigenvector can also be calculated by the fact that eigenvectors of a symmetric matrix are always orthogonal, meaning that if we know that one of them is $\begin{pmatrix}1\\1\end{pmatrix}$, then the other one is $\begin{pmatrix}1\\-1\end{pmatrix}$, since these are orthogonal.

2. Calculate the eigenvalues and eigenvectors of the following matrix!

$$A = \begin{pmatrix}6 & 0 & 2\\0 & -2 & 0\\2 & 0 & 6\end{pmatrix}$$

Solution: Here

$$A - \lambda I = \begin{pmatrix}6 & 0 & 2\\0 & -2 & 0\\2 & 0 & 6\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}6-\lambda & 0 & 2\\0 & -2-\lambda & 0\\2 & 0 & 6-\lambda\end{pmatrix}$$

For the characteristic equation of matrix A, let us use the middle element to calculate the determinant:

$$\det(A - \lambda I) = (-2-\lambda)\begin{vmatrix}6-\lambda & 2\\2 & 6-\lambda\end{vmatrix} = (-2-\lambda)\left[(6-\lambda)^2 - 4\right] = 0.$$

The solutions are $\lambda_1 = -2$, $\lambda_2 = 8$ and $\lambda_3 = 4$, meaning that these are the eigenvalues of matrix A.

Now let us calculate the eigenvectors.

- The eigenvector of $\lambda_1 = -2$: we have to solve the equation $(A - \lambda I)v = (A + 2I)v = 0$, i.e.

$$\begin{pmatrix}8 & 0 & 2\\0 & 0 & 0\\2 & 0 & 8\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

  The first and last equations are independent, meaning that we have to consider both of them:

$$8v_1 + 2v_3 = 0.$$

$$2v_1 + 8v_3 = 0.$$

  Then $v_3 = -4v_1$ and $v_1 = -4v_3$, but these can only hold if $v_1 = v_3 = 0$. However, since the second equation is just an identity, v_2 can be any real number: this means that any vector in the form $\begin{pmatrix}0\\p\\0\end{pmatrix}$ $(p \in \mathbb{R})$ is an eigenvector of λ_1 - e.g. $\begin{pmatrix}0\\1\\0\end{pmatrix}$ is an eigenvector too.

- The eigenvector of $\lambda_2 = 8$: we have to solve the equation $(A - \lambda I)v = (A - 8I)v = 0$, i.e.

$$\begin{pmatrix}-2 & 0 & 2\\0 & -10 & 0\\2 & 0 & -2\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

  The first and last equations are dependent, meaning that it is enough to consider just one of them:

$$-2v_1 + 2v_3 = 0.$$

  Then if $v_1 = p$, then $v_3 = p$ and from the second equation$v_2 = 0$. This means that any vector in the form $\begin{pmatrix}p\\0\\p\end{pmatrix}$ $(p \in \mathbb{R})$ is an eigenvector of λ_1 - e.g. $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is an eigenvector too.

- The eigenvector of $\lambda_3 = 4$: we have to solve the equation $(A - \lambda I)v = (A - 8I)v = 0$, i.e.

$$\begin{pmatrix}2 & 0 & 2\\0 & -6 & 0\\2 & 0 & 2\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

The first and last equations are dependent, meaning that it is enough to consider just one of them:

$$2v_1 + 2v_3 = 0.$$

Then if $v_1 = p$, then $v_3 = -p$ and from the second equation$v_2 = 0$. This means that any vector in the form $\begin{pmatrix} p \\ 0 \\ -p \end{pmatrix}$ ($p \in \mathbb{R}$) is an eigenvector of λ_1 - e.g. $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector too.

Remark: Since A is a symmetric matrix, all the eigenvectors are orthogonal, meaning that if we know two of them, the third one can be calculated: if the third one is $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, then the equations

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = v_1 + v_3 = 0,$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v_2 = 0$$

should hold (since orthogonality means that the scalar product of the vectors is zero), from which we get the vector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

3. Calculate the eigenvalues and eigenvectors of the following matrix!

$$A = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

Solution: Here

$$A - \lambda I = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4-\lambda & -2 & 4 \\ -2 & 1-\lambda & -2 \\ 4 & -2 & 4-\lambda \end{pmatrix}$$

For the characteristic equation of matrix A, let us subtract the last row from the first one:

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & -2 & 4 \\ -2 & 1-\lambda & -2 \\ 4 & -2 & 4-\lambda \end{pmatrix} = \det \begin{pmatrix} -\lambda & 0 & \lambda \\ -2 & 1-\lambda & -2 \\ 4 & -2 & 4-\lambda \end{pmatrix} =$$

$$= -\lambda \left[(1-\lambda)(4-\lambda) - 4 \right] + \lambda \left[4 - 4(1-\lambda) \right] =$$

$$= \lambda(-\lambda^2 + 5\lambda + 4\lambda) = \lambda^2(9-\lambda) = 0.$$

The solutions are $\lambda_1 = 9$ and $\lambda_2 = \lambda_3 = 0$, meaning that these are the eigenvalues of matrix A (here 0 has a multiplicity of 2).

Now let us calculate the eigenvectors.

- The eigenvector of $\lambda_1 = 9$: we have to solve the equation $(A - \lambda I)v = (A - 9I)v = 0$, i.e.

$$\begin{pmatrix} -5 & -2 & 4 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  Let us modify the matrix on the left-hand side by Gaussian elimination: let us add the last row to the first one:

$$\begin{pmatrix} -1 & -4 & -1 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{pmatrix}$$

  Let us multiply the first row by (-1):

$$\begin{pmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{pmatrix}$$

Multiply the first row by 2 and add it to the second one:

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 4 & -2 & -5 \end{pmatrix}$$

Now multiply the first row by 4 and subtract it from the last one:

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & -18 & -9 \end{pmatrix}$$

Multiply the last row by $-1/9$:

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Let us change the order of the last two rows:

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So the equations we have to solve are

$$v_1 + 4v_2 + v_3 = 0,$$

$$2v_2 + v_3 = 0.$$

Let us choose $v_2 = p$: then $v_3 = -2p$ and then $v_1 = -2p$. So e.g. if $p = -1$ then the vector is

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

- The eigenvector of $\lambda_2 = 0$: we have to solve the equation $(A - \lambda I)v = (A - 0I)v = 0$, i.e.

$$\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let us modify the matrix on the left-hand side by Gaussian elimination: let us subtract the first row from the first one:

$$\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us divide the first equation by 2:

$$\begin{pmatrix} 2 & -1 & 2 \\ -2 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, if we subtract the first row from the second one:

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since we have two zero lines, the rank of this matrix is only one, so we will have two free parameters: if $v_2 = p_1$ and $v_3 = p_2$, then $v_1 = \frac{1}{2}(p_1 - 2p_2)$, so any vector in the form $\begin{pmatrix} \frac{1}{2}(p_1 - 2p_2) \\ p_1 \\ p_2 \end{pmatrix}$ will be a solution. For example, if $p_1 = -1$ and $p_2 = 0$, then we get $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. However, since the multiplicity of $\lambda = 0$ is two, it might have two independent eigenvectors: this it the case here, since by the choice $p_1 = 4$ and $p_2 = 1$ we get $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ which is independent form the previous one.

Remark: One can also use the argument mentioned at the end of the previous exercise the calculate the third eigenvector.

4. Calculate the eigenvalues and eigenvectors of the following matrix!

$$A = \begin{pmatrix} 1 & -9 \\ 1 & 1 \end{pmatrix}$$

Solution: Here

$$A - \lambda I = \begin{pmatrix} 1 & -9 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -9 \\ 1 & 1-\lambda \end{pmatrix}$$

The characteristic equation of matrix A is

$$\det(A - \lambda I) = (1-\lambda)^2 + 9 = 0.$$

The solutions are $\lambda_1 = 1 + 3i$ and $\lambda_2 = 1 - 3i$, meaning that these are the eigenvalues of matrix A.

Now let us calculate the eigenvectors.

- The eigenvector of $\lambda_1 = 1 + 3i$: we have to solve the equation $(A - \lambda I)v = (A - (1 + 3i)I)v = 0$, i.e.

$$\begin{pmatrix} -3i & -9 \\ 1 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These equations are always dependent, meaning that it is enough to just consider one of them, e.g. the first one:

$$-3iv_1 - 9v_2 = 0.$$

Let us choose e.g. $v_2 = p$, then $v_1 = 3ip$, so any vector in the form $\begin{pmatrix} 3ip \\ p \end{pmatrix}$ $(p \in \mathbb{R})$ is an eigenvector of λ_1 - e.g. $\begin{pmatrix} 3i \\ 1 \end{pmatrix}$ is an eigenvector too.

- The eigenvector of $\lambda_2 = 1 - 3i$: we have to solve the equation $(A - \lambda I)v = (A - (1 - 3i)I)v = 0$, i.e.

$$\begin{pmatrix} 3i & -9 \\ 1 & 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These equations are always dependent, meaning that it is enough to just consider one of them, e.g. the first one:

$$3iv_1 - 9v_2 = 0.$$

Let us choose e.g. $v_2 = p$, then $v_2 = -3ip$, so any vector in the form $\begin{pmatrix} -3ip \\ p \end{pmatrix}$ $(p \in \mathbb{R})$ is an eigenvector of λ_1 - e.g. $\begin{pmatrix} -3i \\ 1 \end{pmatrix}$ is an eigenvector too.

5. Calculate the eigenvalues and eigenvectors of the following matrix!

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & -2 & 0 \\ -2 & 0 & 6 \end{pmatrix}$$

Solution: Here

$$A - \lambda I = \begin{pmatrix} 6 & 0 & 2 \\ 0 & -2 & 0 \\ -2 & 0 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6-\lambda & 0 & 2 \\ 0 & -2-\lambda & 0 \\ -2 & 0 & 6-\lambda \end{pmatrix}$$

For the characteristic equation of matrix A, let us use the middle element to calculate the determinant:

$$\det(A - \lambda I) = (-2-\lambda) \begin{vmatrix} 6-\lambda & 2 \\ -2 & 6-\lambda \end{vmatrix} = (-2-\lambda)\left[(6-\lambda)^2 + 4\right] = 0.$$

The solutions are $\lambda_1 = -2$, $\lambda_2 = 6 + 2i$ and $\lambda_3 = 6 - 2i$, meaning that these are the eigenvalues of matrix A.

Now let us calculate the eigenvectors.

- The eigenvector of $\lambda_1 = -2$: we have to solve the equation $(A - \lambda I)v = (A + 2I)v = 0$, i.e.

$$\begin{pmatrix} 8 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first and last equations are independent, meaning that we have to consider both of them:

$$8v_1 + 2v_3 = 0.$$

$$-2v_1 + 8v_3 = 0.$$

Then $v_3 = 4v_1$ and $v_1 = -4v_3$, but these can only hold if $v_1 = v_3 = 0$. However, since the second equation is just an identity, v_2 can be any real number: this means that any vector in the form $\begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}$ $(p \in \mathbb{R})$ is an eigenvector of λ_1 - e.g. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector too.

- The eigenvector of $\lambda_2 = 6 + 2i$: we have to solve the equation $(A - \lambda I)v = (A - (6 + 2i)I)v = 0$, i.e.

$$\begin{pmatrix} -2i & 0 & 2 \\ 0 & -8 - 2i & 0 \\ -2 & 0 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first and last equations are dependent, meaning that it is enough to consider just one of them:

$$-2iv_1 + 2v_3 = 0.$$

Then if $v_3 = p$, then $v_1 = -ip$ and from the second equation$v_2 = 0$. This means that any vector in the form $\begin{pmatrix} (-i)p \\ 0 \\ p \end{pmatrix}$ $(p \in \mathbb{R})$ is an eigenvector of λ_1 - e.g. $\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector too.

- The eigenvector of $\lambda_3 = 6 - 2i$: we have to solve the equation $(A - \lambda I)v = (A - (6 - 2i)I)v = 0$. However, if we have a symmetric matrix and complex eigenvectors, then the eigenvectors will be conjugates of each other (so every element is the conjugate of the corresponding element of the other vector), meaning that if the eigenvector in the previous case was $\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$, then in this case it will be $\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$.