Math G2 Practices 5 & 6

Numerical series

1 Partial sums

1. Determine the convergence of the following series by using the definition of convergence!

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution: By definition, a series is convergent if and only if the sequence of the partial sums $S_N = \sum_{n=1}^{N} a_n$ is convergent, i.e.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\sum_{n=1}^{N} a_n \right)$$

is convergent.

In this case

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}.$$

Let us use the identity $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (it can be proved easily by transforming the two fractions into one with a common denominator), and then

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

This is a telescopic sum (a lot of term cancel out each other), meaning that in the end we have

$$S_N = 1 - \frac{1}{N+1}.$$

Then, by definition,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1,$$

so this is a convergent series and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Remark: Another way to prove this claim is to consider the remainder term of the series, i.e.

$$\sum_{n=N+1}^{\infty} a_n.$$

Then, this tends to zero, since

$$\sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = 1 - \left(1 - \frac{1}{N+1}\right) \to 0.$$

so the sum of N-many elements gets closer and closer to the final limit.

2 Geometric series

Geometric series are in the form

$$\sum_{n=0}^{\infty} q^n.$$

A geometric series is convergent, if $|q| < 1$, and then

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

2. Calculate the following series, where $x \in \mathbb{R}$ is a parameter:

$$\sum_{n=0}^{\infty} \left(\frac{x+1}{2x}\right)^n.$$

Solution: this is a geometric series with $q = \dfrac{x+1}{2x}$, so it is only convergent if

$$|q| = \left|\frac{x+1}{2x}\right| < 1,$$

meaning that

$$|x+1| < 2|x|.$$

The easiest way to solve this is to plot these two functions. Then, we have to calculate the two intersections:

$$\begin{aligned} x+1 = -2x \qquad &\longrightarrow \qquad x = -\frac{1}{3}, \\ x+1 = 2x \qquad &\longrightarrow \qquad x = 1. \end{aligned}$$

Then, it is clear that the series is convergent if $x < -\dfrac{1}{3}$ or if $x > 1$. Also, the sum is

$$\sum_{n=0}^{\infty} \left(\frac{x+1}{2x}\right)^n = \frac{1}{1 - \dfrac{x+1}{2x}} = \frac{2x}{x-1}.$$

3 Alternating series

Alternating series are in the form

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

where $a_n > 0$.

Theorem (Leibniz criterion): If for an alternating series we have $a_n \to 0$ and a_n monotonically decreases, then it is called Leibniz series and it is convergent.

3. Determine the convergence of the following series!

$$0.1 - 0.01 + 0.001 - 0.0001 + \ldots$$

Solution: First let us write this series into a compact form, namely

$$\sum_{n=0}^{\infty} (-1)^n 10^{-n} 10^{-1}.$$

Here $a_n = 10^{-n}10^{-1}$, which tends to zero as $n \to \infty$ and it is also monotonically decreasing, meaning that it is a Leibniz series and it is convergent.

Remark: The limit can be calculated with a software: it is going to be 0.09090909....

4. Determine the convergence of the following series!

$$-0.11 + 0.101 - 0.1001 + 0.10001 - 1.100001 + \ldots$$

Solution: First let us write this series into a compact form, namely

$$\sum_{n=0}^{\infty} (-1)^{n+1} \left(0.1 + 10^{-(n+2)}\right).$$

Here $a_n = 0.1 + 10^{-(n+2)}$, which tends to 0.1 as $n \to \infty$, so it is not convergent.

4 Positive series

Positive series are series which have only positive terms in them. There are different methods to determine the convergence of such methods.

- **Ratio criterion:** Let us consider $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$. Then, if this value is smaller than one, then the series is convergent, and if it is bigger than one, then it is divergent.

  This method is usually used when we have a factorial in the exercise.

  5. Determine the convergence of the following series!

  $$\sum_{n=0}^{\infty} \frac{3^n}{n!}.$$

  Solution: Here $a_n = \dfrac{3^n}{n!}$, so

  $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} =$$

  $$= \lim_{n\to\infty} \frac{3\cdot 3^n}{3^n} \frac{n!}{(n+1)\cdot n!} = \lim_{n\to\infty} 3 \cdot \frac{1}{n+1} = 0 < 1,$$

  so since the limit is smaller than one, it converges.

  Remark: The limit of this series is approximately 20.086 (it is actually e^3).

- **Root criterion:** Let us consider $\lim_{n\to\infty} \sqrt[n]{a_n}$. Then, if this value is smaller than one, then the series is convergent, and if it is bigger than one, then it is divergent.

  This method is usually used when we have an nth power in the exercise.

  6. Determine the convergence of the following series!

  $$\sum_{n=1}^{\infty} \frac{3^n}{n^n}.$$

  Solution: Here $a_n = \dfrac{3^n}{n^n}$, so

  $$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^n}} = \lim_{n\to\infty} \frac{3}{n} = 0 < 1,$$

  so since the limit is smaller than one, it converges.

  Remark: The limit of this series is approximately 6.6629.

- **Integral criterion:** Let us define a positive and monotonically decreasing function $f(x)$ from a_n in a way that we replace n by x. Then, if $\int_{x_0}^{\infty} f(x)dx$ is smaller than infinity (where x_0 is the starting index of the series), then the original series converges. Similarly, if $\int_{x_0}^{\infty} f(x)dx$ is infinite, then the initial series diverges.

  This method is usually used when no other method can be used (usually these exercises involve some logarithms).

  7. Determine the convergence of the following series!

  $$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}.$$

  Solution: Here the function will have the form

  $$f(x) = \frac{1}{x(\ln(x))^2}.$$

This is a positive function for $x > 2$, and it is also monotonically decreasing, since $\frac{1}{x}$ decreases and $\ln(x)$ increases, so $(\ln(x))^2$ is also increasing and then $\frac{1}{(\ln(x))^2}$ is decreasing.

Therefore, we can use the previous theorem: for this, we need to calculate the following integral:

$$\int_2^\infty \frac{1}{x(\ln(x))^2}dx = \lim_{c\to\infty}\int_2^c \frac{1}{x(\ln(x))^2}dx = \lim_{c\to\infty}\int_2^c \frac{1}{x}(\ln(x))^{-2}dx =$$

Now we use the integral formula (where $\alpha \neq 0$)

$$\int f'(x)(f(x))^\alpha = \frac{(f(x))^{\alpha+1}}{\alpha+1}.$$

In our case $f(x) = \ln(x)$ (since then $f'(x) = \frac{1}{x}$), $\alpha = -2$, so by continuing the previous integral we get

$$\lim_{c\to\infty}\left[\frac{(ln(x))^{-1}}{-1}\right]_2^c = \lim_{c\to\infty}\left[\frac{-1}{\ln(c)} + \frac{1}{\ln(2)}\right] = \frac{1}{\ln(2)} < \infty,$$

meaning that the initial series also converges.

Remark: The limit of this series is approximately 2.1097.

- **Majorant/minorant criterion**

  Majorant criterion: If there is a convergent series $\sum_{n=1}^\infty b_n$ s.t. $a_n \leq b_n$, then $\sum_{n=1}^\infty a_n$ is also a convergent one.

  Minorant criterion: If there is a divergent series $\sum_{n=1}^\infty c_n = \infty$ s.t. $a_n \geq c_n$, then $\sum_{n=1}^\infty a_n = \infty$ is also a divergent one.

  In these cases the first step is to somehow get a feeling whether the series is convergent or not, and then bound the series either from above by a convergent series, or from below by a divergent one. For this, the series we are going to use are:

  - The series $\sum_{n=1}^\infty \frac{1}{n}$ diverges.
  - The series $\sum_{n=1}^\infty \frac{1}{n^c}$ converges if $c > 1$, e.g. if $c = 2$ then $\sum_{n=1}^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ converges (actually, the limit is $\frac{\pi^2}{6}$).

8. Determine the convergence of the following series!

$$\sum_{n=0}^\infty \frac{1}{n^2+50}$$

Solution: Since the series behaves like $\sum_{n=0}^\infty \frac{1}{n^2}$ for large values of n, we should use the majorant criterion, i.e. have an upper bound for our series.

Since $n^2 + 50 > n^2$, then $\frac{1}{n^2+50} < \frac{1}{n^2}$, meaning that the series $\sum_{n=1}^\infty \frac{1}{n^2}$ is a majorant, which means that the original series is also convergent. (The limit is ≈ 0.2321.)

9. Determine the convergence of the following series!

$$\sum_{n=0}^\infty \frac{\sin^2(n)}{n(n+1)}$$

Solution: Since the series behaves like $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n^2+n)}$ for large values of n, which bevahes like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and it is convergent, we should use the majorant criterion, i.e. have an upper bound for our series.

Since $\dfrac{\sin^2(n)}{n^2+n} < \dfrac{1}{n^2+n}$, meaning that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is a majorant, but it is convergent by Exercise 1, which means that the original series is also convergent. (The limit is ≈ 0.6281.)

10. Determine the convergence of the following series!

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

Solution: Since the series behaves like $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ for large values of n, which is divergent, we should use the minorant criterion, i.e. have a lower bound for our series.

Since $\dfrac{1}{\sqrt{n}} > \dfrac{1}{n}$, meaning that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a minorant, but it is divergent, which means that the original series is also divergent.

11. Determine the convergence of the following series!

$$\sum_{n=0}^{\infty} \frac{8}{5^n+1}$$

Solution: Since the series behaves like $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$ for large values of n, which is convergent (since it is a geometric series), we should use the majorant criterion, i.e. have an upper bound for our series.

Since $\dfrac{8}{5^n+1} < \dfrac{8}{5^n}$, meaning that the series $\displaystyle\sum_{n=1}^{\infty} \frac{8}{5^n}$ is a majorant, but it is convergent (since it is $8 \cdot \displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$), which means that the original series is also convergent. (The limit is ≈ 5.721.)

5 Various exercises

12. Determine the convergence of the following series!

$$\frac{1}{3} + \frac{2!}{3^2} + \frac{3!}{3^3} + \frac{4!}{3^4} + \ldots$$

Solution: This series can be written in the compact form

$$\sum_{n=1}^{\infty} \frac{n!}{3^n}.$$

Since it has a factorial, we suspect that the ration criterion should be used. Here $a_n = \dfrac{n!}{3^n}$, so the limit is

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \frac{3^n}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} (n+1)\frac{1}{3} = \infty,$$

where we used that $(n+1)! = (n+1)n!$ and $3^{n+1} = 3 \cdot 3^n$. Then, since the limit is bigger than 1, the original series is divergent.

12. Determine the convergence of the following series!

$$\sum_{n=3}^{\infty} \frac{\binom{n}{1}}{\binom{n}{3}}.$$

Solution: First let us write this series in a more simple form:

$$\sum_{n=3}^{\infty} \frac{\binom{n}{1}}{\binom{n}{3}} = \sum_{n=3}^{\infty} \frac{\frac{n!}{(n-1)!\cdot 1!}}{\frac{n!}{(n-3)!\cdot 3!}} = \sum_{n=3}^{\infty} \frac{n!}{(n-1)!\cdot 1!} \frac{(n-3)!\cdot 3!}{n!} = \sum_{n=3}^{\infty} \frac{6}{(n-1)(n-2)},$$

where we used the fact that $3! = 1\cdot 2\cdot 3 = 6$ and $(n-1)! = (n-1)(n-2)(n-3)!$.
The series does not have a factorial, an nth power or a logarithm - because of this, we should use one of the majorant or minorant criteria. Since the series behaves like $\sum_{n=1}^{\infty} \frac{1}{n^2}$ for large values of n, we suspect that it is a convergent one, so we should search for a lower bound.
The majorant is

$$\frac{6}{(n-1)(n-2)} < \frac{6}{(n-1)^2} < \frac{6}{n^2},$$

for which the series $\sum_{n=1}^{\infty} \frac{6}{n^2}$ is convergent, so the original series is also convergent. (The limit is 6.)

12. Determine the convergence of the following series!

$$\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}.$$

Solution: Since the series has a logarithm, we might suspect to use the integral criterion. For this, let us define the function

$$f(x) = \frac{1}{x\ln(x)}.$$

This is a positive function, and it is monotone decreasing, since both $\frac{1}{x}$ and $\frac{1}{\ln(x)}$ are decreasing functions.
Then, let us calculate the integral:

$$\int_3^{\infty} \frac{1}{x\ln(x)}dx = \lim_{c\to\infty} \int_3^{c} \frac{1}{x\ln(x)}dx = \lim_{c\to\infty} \int_3^{c} \frac{\frac{1}{x}}{\ln(x)}dx =$$

Then, we can use the integral formula with $f(x) = \ln(x)$

$$\int \frac{f'(x)}{f(x)}dx = \ln(|f(x)|) + c,$$

which means that

$$= \lim_{c\to\infty} [\ln(|\ln(x)|)]_3^c =$$

Since here $x > 1$, the absolute value can be omitted:

$$\lim_{c\to\infty} [\ln(\ln(x))]_3^c = \lim_{c\to\infty} [\ln(\ln(c)) - \ln(\ln(3))]$$

Since $\lim_{c\to\infty} \ln(c) = \infty$, then $\lim_{c\to\infty} \ln(\ln(c)) = \infty$, so the limit is infinite. Then, by the theorem the original series is also infinite, so it diverges.

15. Determine the convergence of the following series!

$$\sum_{n=3}^{\infty} \left(\frac{n}{n^2+1}\right)^{n^2}.$$

Solution: Since we have an nth power, we should use the root criterion. Here $a_n = \left(\frac{n}{n^2+1}\right)^{n^2}$, meaning that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n^2+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n^2+1}\right)^{n} = \lim_{n\to\infty} \left(\frac{1}{\frac{n^2+1}{n}}\right)^{n} = \lim_{n\to\infty} \frac{1}{\left(n+\frac{1}{n}\right)^n}$$

The problem here is we do not really know the limit of this sequence. However, if we would have $1+\dfrac{1}{n}$ inside the brackets, then we would know the limit. This would be true if in the original series we would have $\dfrac{n}{n+1}$ instead of $\dfrac{n}{n^2+1}$.
It is clear that
$$\frac{n}{n^2+1} < \frac{n}{n+1},$$
and then
$$\left(\frac{n}{n^2+1}\right)^{n^2} < \left(\frac{n}{n+1}\right)^{n^2},$$
meaning that this is a majorant. If we apply the root condition to this majorant, we get
$$\lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty} \left(\frac{1}{\frac{n+1}{n}}\right)^{n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$
Since this is smaller than one, the majorant converges, so the original series is a convergent one too.

16. Determine the convergence of the following series!
$$\sum_{n=3}^{\infty} \left(\frac{n}{n-1}\right)^n \frac{1}{2^n}.$$

Solution: Since we have an nth power, we might use the root criterion. Here $a_n = \left(\frac{n}{n-1}\right)^n \dfrac{1}{2^n}$, so the limit is
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n-1}\right)^n \frac{1}{2^n}} = \lim_{n\to\infty} \frac{n}{n-1}\frac{1}{2} = \lim_{n\to\infty} \frac{n}{2n-2} = \lim_{n\to\infty} \frac{1}{2-\frac{2}{n}} = \frac{1}{2} < 1,$$
so the original series converges.

17. Determine the convergence of the following series!
$$\sum_{n=3}^{\infty} \left(\frac{n}{n-1}\right)^{n^2} \frac{1}{2^n}.$$

Solution: Since we have an nth power, we might use the root criterion. Here $a_n = \left(\frac{n^2}{n-1}\right)^n \dfrac{1}{2^n}$, so the limit is
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n-1}\right)^{n^2} \frac{1}{2^n}} = \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n \frac{1}{2} = \lim_{n\to\infty} \left(\frac{n-1+1}{n-1}\right)^n \frac{1}{2} =$$
$$= \lim_{n\to\infty} \left(1+\frac{1}{n-1}\right)^n \frac{1}{2} = \lim_{n\to\infty} \left(1+\frac{1}{n-1}\right)^{n-1} \left(1+\frac{1}{n-1}\right)\frac{1}{2} = e\cdot 1\cdot \frac{1}{2} > 1$$
so the original series diverges.

18. Determine the convergence of the following series!
$$\sum_{n=3}^{\infty} \frac{(\arctan(n))^n}{(n+1)2^{n-1}}.$$

Solution: Since we have an nth power, we might use the root criterion. Here $a_n = \dfrac{(\arctan(n))^n}{(n+1)2^{n-1}}$, so the limit is
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(\arctan(n))^n}{(n+1)2^{n-1}}} = \lim_{n\to\infty} \frac{\arctan(n)}{\sqrt[n]{n+1}\sqrt[n]{\frac{2^n}{2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{2}\arctan(n)}{\sqrt[n]{n+1}\cdot 2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4} < 1,$$
where we used the fact that $\sqrt[n]{2} \to 1$, $\sqrt[n]{n+1} \to 1$ (the latter one can be seen by the squeeze theorem) and $\lim_{n\to\infty} \arctan(n) = \dfrac{\pi}{2}$. Since the limit is smaller than one, the original series converges (the limit is ≈ 0.9649).

6 Absolute and conditional convergence

We say that a series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**, if the series $\sum_{n=1}^{\infty} |a_n|$ is also convergent.

Proposition: If a series is absolutely convergent, then it is convergent.

If a series is convergent but not absolutely convergent, then it is called **conditionally convergent**.

19. For which values of k is the following series absolutely convergent or conditionally convergent?

$$\sum_{n=1}^{\infty}(-1)^n n^k.$$

Solution: For the absolute convergence, the series we have to observe is $\displaystyle\sum_{n=1}^{\infty} n^k$.

- If $k \geq 0$, then the sequence n^k does not converge to zero, so it cannot be convergent (a necessary condition for convergence is that the sequence of the terms should tend to zero).
- If $-1 < k < 0$, then it is divergent. For this, let us use the integral criterion, i.e. consider the function $f(x) = x^k$ where $-1 < k < 0$. Then,

$$\int_1^{\infty} x^k dx = \lim_{c\to\infty} \int_1^c x^k dx = \lim_{c\to\infty} \left[\frac{x^{k+1}}{k+1}\right]_1^c = \lim_{c\to\infty} \left[\frac{c^{k+1}}{k+1} - \frac{1}{k+1}\right] = \infty,$$

  since here $0 < k+1 < 1$. Then, the original series is divergent.
- If $k = -1$, then it is also divergent. For this, let us use the integral criterion, i.e. consider the function $f(x) = \dfrac{1}{x}$. Then,

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{c\to\infty} \int_1^c \frac{1}{x} dx = \lim_{c\to\infty} [\ln(x)]_1^c = \lim_{c\to\infty} [\ln(c) - \ln(1)] = \infty.$$

  Then, the original series is divergent.
- If $-1 > k$, then it is convergent. For this, let us use the integral criterion, i.e. consider the function $f(x) = x^k$ where $-1 > k$. Then,

$$\int_1^{\infty} x^k dx = \lim_{c\to\infty} \int_1^c x^k dx = \lim_{c\to\infty} \left[\frac{x^{k+1}}{k+1}\right]_1^c = \lim_{c\to\infty} \left[\frac{c^{k+1}}{k+1} - \frac{1}{k+1}\right] =$$

$$= \lim_{c\to\infty} \left[\frac{c^{k+1}}{k+1}\right] - \frac{1}{k+1} = -\frac{1}{k+1} < \infty,$$

  since here $k+1 < 0$. Then, the original series is convergent.

To sum it up, the series is absolutely convergent if $k < -1$ and consequently it is convergent for $k < -1$.

What can we say about conditional convergence? For this, we have to observe the convergence of the original series for $k > -1$.

The original series is an alternating series, consequently we have to use the Leibniz criterion. It can be seen that if $k < 0$, then the sequence n^k is monotonically decreasing, and it also tends to zero - because of this, by the Leibniz criterion the original series is conditionally convergent if $-1 < k < 0$.

In conclusion, we can say that it is absolutely convergent when $k < -1$, it is conditionally convergent if $-1 \leq k \leq 0$ and it is divergent if $k > 0$.

7 Error estimation

In practice we cannot calculate the sum of infinitely many elements. Because of this, the thing we can calculate is $\sum_{n=1}^{N} a_n$, i.e. the sum of the first N-many elements. We would like to have an estimate for the error we have in this case, i.e. for the value

$$e_N = \left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \right|.$$

- If we have a Leibniz series, then $e_N < |a_{N+1}|$.
- If we have a positive series, then we should either approximate the remaining terms by a geometric series, or by some integral (see the next Exercises).

20. Give an approximation for the error if we approximate the following series by S_4!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Solution: This is a Leibniz series (since $a_n = \dfrac{1}{n^2}$ is a monotone decreasing sequence and it tends to zero), so (since $N = 4$) the error is

$$e_4 < |a_5| = \frac{1}{5^2},$$

so the error is at most 0.04 in this case.

Remark: The real error is ≈ 0.02386.

21. Give an approximation for the error if we approximate the following series by S_3!

$$\sum_{n=1}^{\infty} \frac{1}{(2n)!}$$

Solution: This is a positive series, so we should bound the remaining terms, i.e. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{(2n)!}$ by a geometric series.

$$\left| \sum_{n=4}^{\infty} \frac{1}{(2n)!} \right| = \sum_{n=4}^{\infty} \frac{1}{(2n)!} = \frac{1}{8!} + \frac{1}{10!} + \frac{1}{12!} + \dots = \frac{1}{8!}\left(1 + \frac{1}{9\cdot 10} + \frac{1}{9\cdot 10\cdot 11\cdot 12} + \dots\right) \leq$$

$$\leq \frac{1}{8!}\left(1 + \frac{1}{9\cdot 9} + \frac{1}{9\cdot 9\cdot 9\cdot 9} + \dots\right) = \frac{1}{8!}\left(\sum_{n=0}^{\infty}\left(\frac{1}{9^2}\right)^n\right) = \frac{1}{8!}\frac{1}{1-\frac{1}{9^2}} \approx 2.51\cdot 10^{-5},$$

where first we changed every number in the denominators to nines, and then we used the formula for the sum of a geometric series. Therefore, the error is at most $2.51 \cdot 10^{-5}$.

Remark: The real error is $\approx 2.5079 \cdot 10^{-5}$

22. Give an approximation for the error if we approximate the following series by S_4!

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Solution: The problem here is that since the denominator gets bigger and bigger as k goes to infinity, we cannot give an upper bound like in the previous case by using a geometric series. Instead of this, we will bound the remaining terms by some integrals:

$$\int_4^{\infty} \frac{1}{(1+x)^4}dx \leq \sum_{n=5}^{\infty} \frac{1}{n^4} \leq \int_4^{\infty} \frac{1}{x^4}dx$$

These integrals can be calculated:

$$\int_4^{\infty} \frac{1}{(1+x)^4}dx = \lim_{c\to\infty} \int_4^{c} \frac{1}{(1+x)^4}dx = \lim_{c\to\infty}\left[-\frac{1}{3}\frac{1}{(1+x)^3}\right]_4^c = \frac{1}{3}\cdot\frac{1}{5^3} \approx 0.00267,$$

$$\int_4^{\infty} \frac{1}{x^4}dx = \lim_{c\to\infty} \int_4^{c} \frac{1}{x^4}dx = \lim_{c\to\infty}\left[-\frac{1}{3}\frac{1}{x^3}\right]_4^c = \frac{1}{3}\cdot\frac{1}{4^3} \approx 0.0052083$$

So an upper bound for the error is 0.0052083.

Remark: The real error is 0.003571

23. How many elements should we add up of this series such that our error is smaller than $\varepsilon = 10^{-2}$?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$$

Solution: The question can be rephrased as: what is the first value of N for which we have $e_N < 10^{-2}$?

This is a Leibniz series (since $\frac{1}{n!}$ tends to zero and it is decreasing), so if we add up N-many elements then the error is

$$e_N = |a_{N+1}| = \frac{1}{(N+1)!}$$

The first value for which this is smaller than $\frac{1}{100}$ is $n = 4$, so we should add up the first four elements and then the error is smaller than 0.01.

Remark: The real error here is 0.00712.

24. How many elements should we add up of this series such that our error is smaller than $\varepsilon = 10^{-2}$?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solution: This is a positive series, but since the denominator gets larger and larger as n increases, we cannot bound it from above by a geometric series. Instead of this, we can use the integral technique discussed in Exercise 22.:

$$\sum_{N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{1}{x^2} dx = \lim_{c \to \infty} \left[-\frac{1}{x} \right]_N^c = \frac{1}{N}$$

This should be smaller than $\frac{1}{100}$: in this case $N = 101$ is a proper choice.

Remark: The real error here is 0.00985.