Math G2 Practice 7

Function series, Power series, Taylor series

1 Function series

A series of a functions is defines as the infinite sum

$$\sum_{k=0}^{\infty} f_k(x) = f_0(x) + f_1(x) + f_2(x) + \dots$$

The **interval of convergence** is the interval of those points for which the above series is a convergent one.

1. Determine the interval of convergence of the following series, and calculate the sum in these points!

$$\sum_{n=0}^{\infty} (\ln(x))^n.$$

Solution: For a fixed value of x, this is a geometric series with $q = \ln(x)$, so it is convergent if $|q| = |\ln(x)| < 1$, meaning that $\frac{1}{e} \leq x \leq e$.

The sum at these points can be given by the usual formula for the sum of a geometric series:

$$\frac{1}{1-q} = \frac{1}{1-\ln(x)}.$$

2. Determine the interval of convergence of the following series! Where is it absolutely convergent?

$$\sum_{n=0}^{\infty} \frac{(1-x)x^n}{n}.$$

Solution: For the absolute convergence we have to observe the series

$$\sum_{n=0}^{\infty} \frac{|1-x||x|^n}{n}.$$

By the root criterion,

$$\lim_{n\to\infty} \sqrt[n]{\frac{|1-x||x|^n}{n}} = \lim_{n\to\infty} |x| \frac{\sqrt[n]{|1-x|}}{\sqrt[n]{n}}.$$

This tends to $|x|$ if $x \neq 1$ and to 0 if $x = 1$, meaning that it is absolutely convergent when $|x| < 1$ (since the limit should be smaller than one).

Let us observe the endpoints of the interval $(-1, 1)$: for $x = 1$, we get an all-zero sum, so it is absolutely convergent there. For $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{2(-1)^n}{n},$$

which is not absolutely convergent but it is convergent. Consequently, the series is absolutely convergent for $x \in (-1, 1]$ and it is convergent for $[-1, 1]$.

3. Determine the interval of convergence of the following series! Where is it uniformly convergent?

$$\sum_{n=0}^{\infty} \frac{\cos(nx)}{n^2 + x^2}.$$

Solution: By the Theorem of Weierstrass, if we an find such a sequence that after some time $|f(x)| \leq a_n$ for all values of x, and $\sum_{n=0}^{\infty} a_n$ is convergent, then the series $\sum_{n=0}^{\infty} f(x)$ is uniformly and absolutely convergent.

In this case we have

$$\left|\frac{\cos(nx)}{n^2+x^2}\right| \leq \frac{|\cos(nx)|}{n^2+x^2} \leq \frac{1}{n^2+x^2} \leq \frac{1}{n^2}.$$

Then, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the original series is also uniformly and absolutely convergent.

2 Power series

Power series are special function series in the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

Here a_n are real numbers (they are the coefficients) and x_0 is a given value, usually referred to as the center of the convergence.

The name of the latter one comes from the fact that the interval of convergence for a power series is always in the form $(x_0 - R, x_0 + R)$, where R is a non-negative real value, but it can also be infinity. If $R = 0$, then the series is convergent only for $x = x_0$, and if $R = \infty$ then the series is convergent for all values of $x \in \mathbb{R}$.

It is worth note mentioning that the endpoints $x_0 - R$ and $x_0 + R$ might be inside the interval of convergence, or they might be not, it depends on the given example, so these two points should always be examined separately.

The value of R can be calculated by two different methods:

- **Root criterion:** Let us consider the limit $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then, if α is a non-zero finite value, the $R = \frac{1}{\alpha}$. If $\alpha = 0$, then $R = \infty$, and if $\alpha = \infty$, then $R = 0$.
- **Ratio criterion:** Let us consider the limit $\alpha = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. Then, if α is a non-zero finite value, the $R = \frac{1}{\alpha}$. If $\alpha = 0$, then $R = \infty$, and if $\alpha = \infty$, then $R = 0$.

4. Let us determine the interval of convergence of the following power series!

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$$

Solution: The series can be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(x-2)^2,$$

which means that here $a_n = \frac{1}{n^2}$ and $x_0 = 2$. We can use both of the criteria in this case:

- *Root criterion:* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \limsup_{n\to\infty} \left(\frac{1}{\sqrt[n]{n}}\right)^2 = 1$, so $R = \frac{1}{1} = 1$.
- *Ration criterion:* $\alpha = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1$, meaning that $R = \frac{1}{1} = 1$.

Then, the interval of convergence is $(2-1, 2+1) = (1, 3)$.

We should observe the endpoints $x_1 = 1$ and $x_2 = 3$ separately.

- If $x = 1$, then the series has the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which can be proved to be a Leibniz series, so it is convergent.

- If $x = 1$, then the series has the form

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is convergent.

Then, the interval of convergence is $[1, 3]$.

5. Let us determine the interval of convergence of the following power series!

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}.$$

Solution: The series can be rewritten as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x-0)^2,$$

which means that here $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ and $x_0 = 0$. We are going to use the ratio criterion in this case:

$$\alpha = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1,$$

meaning that $R = \dfrac{1}{1} = 1$.
Then, the interval of convergence is $(0-1, 0+1) = (-1, 1)$.
We should observe the endpoints $x_1 = -1$ and $x_2 = 1$ separately.

- If $x = -1$, then the series has the form

$$\sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which is a divergent series (see Exercise 10. from the previous exercise sheet).

- If $x = 1$, then the series has the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

which is a Leibniz series so it is convergent.

Then, the interval of convergence is $(-1, 1]$.

3 Taylor series

Taylor series are special power series in the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

where $f^{(n)}(x_0)$ denotes the nth derivative of function f evaluated at point x_0.

In the following exercises, we will not use this formula, but we are going to use the Taylor series of some well-known functions around $x_0 = 0$ (here $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$).

- $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$, interval of convergence: $\mathbb{R}$.
- $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$, interval of convergence: $\mathbb{R}$.
- $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$, interval of convergence: $\mathbb{R}$.
- $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$, interval of convergence: $\mathbb{R}$.
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$, interval of convergence: $\mathbb{R}$.
- $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$, interval of convergence: $|x| < 1$.
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$, interval of convergence: $|x| < 1$.
- If n is not a positive integer, then

  $$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \binom{n}{5}x^5 + \ldots,$$

  interval of convergence: $|x| < 1$. Here $\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!}$, so the numerator has k-many elements.
- $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$, interval of convergence: $|x| < 1$.

6. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = \sin(x)\cos(x).$$

Solution:

$$f(x) = \sin(x)\cos(x) = \frac{1}{2}\sin(2x) =$$

where we used that $\sin(2x) = 2\sin(x)\cos(x)$.

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+1)!}x^{2n+1}.$$

The interval of convergence is $x \in \mathbb{R}$.

Alternatively, one can also calculate the derivatives of the function $\sin(x)\cos(x)$ at $x = 0$ and then use the definition.

7. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = \sin^2(x).$$

Solution:

$$f(x) = \sin^2(x) = \frac{1-\cos(2x)}{2} =$$

where we used the facts $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin^2(x) + \cos^2(x) = 1$.

$$= \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!} = \frac{1}{2} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{2n-1}}{(2n)!}x^{2n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}}{(2n)!}x^{2n},$$

where we used the fact that the $n = 0$ term of the right sum was $-\frac{1}{2}$, so it makes the other $\frac{1}{2}$ vanish. The interval of convergence is $x \in \mathbb{R}$.

Alternatively, one can also calculate the derivatives of the function $\sin^2(x)$ at $x = 0$ and then use the definition.

8. Determine the Taylor series of the following function and the corresponding interval of convergence! Here $a \in \mathbb{R}^+$.

$$f(x) = a^x.$$

Solution:

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)} = \sum_{n=0}^{\infty}\frac{(x\ln(a))^n}{n!} = \sum_{n=0}^{\infty}\frac{(\ln(a))^n}{n!}x^n.$$

The interval of convergence is $x \in \mathbb{R}$.

9. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = (1+x)^3.$$

Solution: Since

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$

we do not even have to use the previous formulas, since this can be thought of as a (finite) Taylor series with $a_0 = a_3 = 1$, $a_1 = a_2 = 3$ and $a_n = 0$ for $n > 3$. This of course holds for any $x \in \mathbb{R}$. so the interval of convergence is $\mathbb{R}$.

10. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = (1+x)^{-3}.$$

Solution: Since the power is not a positive integer, we can use the previous formula:

$$(1+x)^{-3} = \sum_{n=0}^{\infty}\binom{-3}{n}x^n.$$

The interval of convergence is $|x| < 1$.

11. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = (1+x)^{-1/3}.$$

Solution: Since the power is not a positive integer, we can use the previous formula:

$$(1+x)^{-1/3} = \sum_{n=0}^{\infty}\binom{-\frac{1}{3}}{n}x^n.$$

The interval of convergence is $|x| < 1$.

12. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = \frac{x}{1+x^2}.$$

Solution: We know that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n,$$

then

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n (x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

and

$$\frac{x}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}.$$

Another method: It can be seen that

$$\frac{x}{1+x^2} = \frac{1}{2}\left(\ln(1+x^2)\right)'.$$

Since we know that

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n},$$

then

$$\ln(1+x^2) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}.$$

Then, if we calculate the derivative:

$$\frac{1}{2}\left(\ln(1+x^2)\right)' = \frac{1}{2}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}\right)' =$$

Since the series is uniformly convergent, then the order of the differentiation and the sum can be changed:

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{x^{2n}}{n}\right)' = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}(2n)\frac{x^{2n-1}}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}x^{2n-1}.$$

13. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right).$$

Solution:

$$\ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \ln\left(\left(\frac{1+x}{1-x}\right)^{1/2}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) =$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-x)^n}{n}\right) =$$

here we use that $(-1)^{n+1}(-1)^n = -1$:

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} + \sum_{n=1}^{\infty}\frac{x^n}{n}\right) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} + \frac{x^n}{n} =$$

when n is even, then $(-1)^{n+1} = -1$, so in this case these terms vanish, meaning that we only have to care about the case when n is odd, i.e. $n = 2k+1$.

$$= \frac{1}{2}\sum_{k=1}^{\infty}2\frac{x^{2k+1}}{n} = \sum_{k=1}^{\infty}\frac{x^{2k+1}}{n}.$$

The interval of convergence is $|x| < 1$.

14. Determine the Taylor series of the following function and the corresponding interval of convergence!

$$f(x) = \arccos(x).$$

Solution: The main idea of this exercise is the fact that

$$\int_0^x (\arccos(t))' dt = \arccos(x) - \arccos(0) = \arccos(x) - \frac{\pi}{2}.$$

Then,

$$\arccos(x) = \frac{\pi}{2} + \int_0^x (\arccos(t))' dt = \frac{\pi}{2} + \int_0^x \frac{-1}{\sqrt{1-t^2}} dt = \frac{\pi}{2} - \int_0^x (1-t^2)^{-1/2} dt =$$

Now we use the Taylor series of the function $(1-t^2)^{-1/2}$:

$$= \frac{\pi}{2} - \int_0^x \left(\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-t^2)^n \right) dt =$$

Since the series is uniformly convergent, wwe can change the order of the integration and the sum:

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\int_0^x \binom{-\frac{1}{2}}{n} (-1)^n t^{2n} dt \right) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \left(\int_0^x t^{2n} dt \right) =$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \left[\frac{t^{2n+1}}{2n+1} \right]_0^x = \frac{\pi}{2} + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

The interval of convergence is $|x| < 1$.

15. Calculate the following integral by using the Taylor series of the function!

$$\int_0^x \frac{\sin(t)}{t} dt.$$

Solution:

$$\int_0^x \frac{\sin(t)}{t} dt = \int_0^x \frac{1}{t} \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \right) dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} \right) dt =$$

Since the series is uniformly convergent, we can change the order of the integral and the sum:

$$= \sum_{n=0}^{\infty} \left(\int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!} dt \right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(\int_0^x t^{2n} dt \right) =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left[\frac{t^{2n+1}}{2n+1} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{2n+1}$$

The interval of convergence is $\mathbb{R}$.

16. Give the Taylor series of the following function around $x_0 = 1$.

$$f(x) = e^x.$$

Solution: The goal here is to have a power series in the form $\sum_{n=0}^{\infty} a_n (x-1)^n$. For this, we would need to have the expression $(x-1)$ in our function.

$$e^x = e^{(x-1)+1} = e \cdot e^{x-1} = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n.$$

The interval of convergence is $\mathbb{R}$.

17. Give the Taylor series of the following function around $x_0 = 1$.

$$f(x) = \ln(x).$$

Solution: The goal here is to have a power series in the form $\sum_{n=0}^{\infty} a_n(x-1)^n$. For this, we would need to have the expression $(x-1)$ in our function.

$$\ln(x) = \ln((x-1)+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$$

The interval of convergence is $|x-1| < 1$.

18. Give the Taylor series of the following function around $x_0 = c$ (where $c \neq -1$).

$$f(x) = \frac{1}{1+x}.$$

Solution: The goal here is to have a power series in the form $\sum_{n=0}^{\infty} a_n(x-c)^n$. For this, we would need to have the expression $(x-c)$ in our function.

$$\frac{1}{1+x} = \frac{1}{1+(x-c)+c} = \frac{1}{(1+c)+(x-c)} = \frac{1}{1+c}\frac{1}{1+\frac{x-c}{1+c}} = \frac{1}{1+c}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-c}{1+c}\right)^n =$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(x-c)^n}{(1+c)^{n+1}}$$

The interval of convergence is those points for which $\left|\frac{x-c}{1+c}\right| < 1$.

4 Application: approximation of a hard integral

19. Approximate the following integral by using the Taylor series of the function:

$$\int_0^1 \sin(x^2)dx.$$

How many terms should we add up in the series that our error is smaller than $5 \cdot 10^{-4}$?

Solution:

$$\int_0^1 \sin(x^2)dx = \int_0^1 \left(\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}\right)dx =$$

Since the series converges uniformly, the order of the integral and the sum can be changed:

$$= \sum_{n=0}^{\infty}\left(\int_0^1 \frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}dx\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\int_0^1 x^{4n+2}dx\right) =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\frac{x^{4n+3}}{4n+3}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(4n+3)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!(4n+3)}$$

This can be proved to be a Leibniz series, so it converges.

The error after the addition of N-many elements is

$$|e_N| < |a_{N+1}| = \frac{1}{(2(N+1)+1)!(4(N+1)+3)} = \frac{1}{(2N+3)!(4N+7)}$$

If $N = 1$, then the above value is $\approx 7.57 \cdot 10^{-4}$, but for $N = 2$ it is $1.32 \cdot 10^{-5}$, so it is smaller than the desired error. This means that we have to calculate the following sum for the integral:

$$\int_0^1 \sin(x^2)dx \approx \sum_0^2(-1)^n\frac{1}{(2n+1)!(4n+3)} = \frac{1}{3} - \frac{1}{3!\cdot 7} + \frac{1}{5!\cdot 11}$$