Math G2 Practices 8

Fourier series

The Fourier series of a function which is periodic by P, $P \in \mathbb{R}$ (or: it is defined on $[0, P]$) is given by

$$(\mathcal{F}f)(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k}{P}x\right) + b_k \sin\left(\frac{2\pi k}{P}x\right),$$

where

$$a_0 = \frac{1}{P}\int_0^P f(x)dx,$$

$$a_k = \frac{2}{P}\int_0^P f(x)\cos\left(\frac{2\pi k}{P}x\right)dx,$$

$$b_k = \frac{2}{P}\int_0^P f(x)\sin\left(\frac{2\pi k}{P}x\right)dx.$$

1. Determine the Fourier series of the following function:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < \pi, \\ -x & \text{if } \pi \le x < 2\pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$.

Solution: Here $P = 2\pi$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\ dx = \frac{1}{2\pi}\left(\int_0^{\pi} x\ dx + \int_{\pi}^{2\pi} -x\ dx\right) = \frac{1}{2\pi}\left(\left[\frac{x^2}{2}\right]_0^{\pi} - \left[\frac{x^2}{2}\right]_{\pi}^{2\pi}\right) =$$

$$= \frac{1}{2\pi}\left(\left[\frac{\pi^2}{2} - 0\right] - \left[\frac{(2\pi)^2}{2} - \frac{\pi^2}{2}\right]\right) = \frac{1}{2\pi}\left(\frac{\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right) = \frac{-\pi^2}{2\pi} = -\frac{\pi}{2}.$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\left(\int_0^{\pi} x\cos(kx)dx + \int_{\pi}^{2\pi} -x\cos(kx)dx\right) = \quad (1)$$

Now we use the fact that

$$\int x\cos(kx)dx = x\frac{\sin(kx)}{k} - \int \frac{\sin(kx)}{k}dx,$$

so the calculations of Equation (1) can be followed as

$$= \frac{1}{\pi}\left(\left[x\frac{\sin(kx)}{k}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(kx)}{k}dx - \left[x\frac{\sin(kx)}{k}\right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\frac{\sin(kx)}{k}dx\right) =$$

$$= \frac{1}{\pi}\left(\left[\pi\frac{\sin(k\pi)}{k}\right] - \int_0^{\pi}\frac{\sin(kx)}{k}dx - \left[2\pi\frac{\sin(2\pi k)}{k} - \pi\frac{\sin(k\pi)}{k}\right] + \int_{\pi}^{2\pi}\frac{\sin(kx)}{k}dx\right) =$$

Now we use the fact that $\sin(k\pi) = 0$ for every integer value of k.

$$= \frac{1}{\pi}\left(-\int_0^{\pi}\frac{\sin(kx)}{k}dx + \int_{\pi}^{2\pi}\frac{\sin(kx)}{k}dx\right) = \frac{1}{\pi}\left(-\left[-\frac{\cos(kx)}{k^2}\right]_0^{\pi} + \left[\frac{-\cos(kx)}{k^2}\right]_{\pi}^{2\pi}\right) =$$

$$= \frac{1}{\pi}\left(-\left[-\frac{\cos(k\pi)}{k^2} + \frac{1}{k^2}\right] + \left[-\frac{\cos(2k\pi)}{k^2} + \frac{\cos(k\pi)}{k^2}\right]\right) =$$

Now we use the fact that $\cos(2k\pi) = 1$ for any value of k:

$$= \frac{1}{\pi}\left(-\left[-\frac{\cos(k\pi)}{k^2} + \frac{1}{k^2}\right] + \left[-\frac{1}{k^2} + \frac{\cos(k\pi)}{k^2}\right]\right) = \frac{1}{\pi}\left(2\frac{\cos(k\pi)}{k^2} - \frac{2}{k^2}\right) =$$

The value of $\cos(k\pi)$ might be 1 if k is even and can be -1 when k is odd, so the value is:

$$= \begin{cases} 0 & \text{if k is even,} \\ -\dfrac{4}{\pi k^2} & \text{if k is odd.} \end{cases}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx = \frac{1}{\pi}\left(\int_0^{\pi} x\sin(kx)dx + \int_{\pi}^{2\pi}(-x)\sin(kx)dx\right) =$$

Now we use the fact that

$$\int x\sin(kx)dx = -x\frac{\cos(kx)}{k} + \int \frac{\cos(kx)}{k}dx,$$

so the previous line continues as

$$= \frac{1}{\pi}\left(\left[-x\frac{\cos(kx)}{k}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(kx)}{k}dx - \left[-x\frac{\cos(kx)}{k}\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\frac{\cos(kx)}{k}dx\right) =$$

$$= \frac{1}{\pi}\left(\left[-\pi\frac{\cos(k\pi)}{k}\right] + \int_0^{\pi}\frac{\cos(kx)}{k}dx - \left[-(2\pi)\frac{\cos(2k\pi)}{k} + \pi\frac{\cos(k\pi)}{k}\right] - \int_{\pi}^{2\pi}\frac{\cos(kx)}{k}dx\right) =$$

Now we use the fact that $\cos(2k\pi) = 1$ for any integer value of k:

$$== \frac{1}{\pi}\left(-\pi\frac{\cos(k\pi)}{k} + \int_0^{\pi}\frac{\cos(kx)}{k}dx - \left[\frac{-2\pi}{k} + \pi\frac{\cos(k\pi)}{k}\right] - \int_{\pi}^{2\pi}\frac{\cos(kx)}{k}dx\right) =$$

$$== \frac{1}{\pi}\left(-\pi\frac{2\cos(k\pi)}{k} + \frac{2\pi}{k} + \int_0^{\pi}\frac{\cos(kx)}{k}dx - \int_{\pi}^{2\pi}\frac{\cos(kx)}{k}dx\right) =$$

$$= \frac{1}{\pi}\left(-\pi\frac{2\cos(k\pi)}{k} + \frac{2\pi}{k} + \left[\frac{\sin(kx)}{k^2}\right]_0^{\pi} - \left[\frac{\sin(kx)}{k^2}\right]_{\pi}^{2\pi}\right) =$$

Now we use the fact that $\sin(kx) = 0$ for any integer value of k.

$$= \frac{1}{\pi}\left(-\pi\frac{2\cos(k\pi)}{k} + \frac{2\pi}{k}\right) =$$

When k is even, then $\cos(kx) = 1$ and when k is odd then $\cos(kx) = -1$.

$$= \begin{cases} 0 & \text{if k is even,} \\ -\dfrac{4}{k} & \text{if k is odd.} \end{cases}$$

Then, the Fourier series has the form

$$(\mathcal{F}f)(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} -\frac{4}{\pi(2k-1)^2}\cos((2k-1)x) + \frac{4}{2k-1}\sin((2k-1)x).$$

The reason for this is that the index should be odd (the even terms are zero).

2. Determine the Fourier series of the function $f(x) = |\sin(x)|$.

   Solution: Here $P = 2\pi$ ($P = \pi$ can also be used). We are going to use the fact that $|\sin(x)| = \sin(x)$ for $x \in (0, \pi)$ and $|\sin(x)| = -\sin(x)$ for $x \in (\pi, 2\pi)$.

$$a_0 = \frac{1}{2\pi}\int_0^{\pi}\sin(x)dx + \int_{\pi}^{2\pi}(-\sin(x))dx = \frac{1}{\pi}\int_0^{\pi}\sin(x)dx =$$

Here we used the fact that the function is symmetric.

$$= \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} = \frac{1}{\pi}(-(-1) - (-1)) = \frac{2}{\pi}.$$

$$a_k = \frac{1}{\pi}\left(\int_0^{\pi}\sin(x)\cos(kx)dx + \int_{\pi}^{2\pi}(-\sin(x))\cos(kx)dx\right) =$$

Now we use symmetry:

$$= \frac{1}{\pi}\left(\int_0^{\pi} \sin(x)\cos(kx)dx + \int_{-\pi}^{0} (-\sin(x))\cos(kx)dx\right) =$$

Then, since the cosine function is even and the sine function is odd:

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(kx)dx =$$

Here we can use either partial integration or the identity

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) - \sin(\beta-\alpha),$$

if we use the latter with $\alpha = x$ and $\beta = kx$, then

$$= \frac{1}{\pi}\int_0^{\pi} \sin((k+1)x) - \sin((k-1)x)dx = \frac{1}{\pi}\left[-\frac{\cos((k+1)x)}{k+1} + \frac{\cos((k-1)x)}{k-1}\right]_0^{\pi} =$$

$$= \frac{1}{\pi}\left[-\frac{\cos((k+1)\pi)}{k+1} + \frac{\cos((k-1)\pi)}{k-1} + \frac{1}{k+1} - \frac{1}{k-1}\right] =$$

now we use the fact that $\cos((k+1)\pi) = \cos((k-1)\pi) = 1$ if k is odd and they are both -1 if k is even:

$$= \begin{cases} 0 & \text{if k is odd,} \\ \frac{2}{\pi}\left(\frac{1}{k+1} - \frac{1}{k-1}\right) & \text{if k is even.} \end{cases}$$

Since f is an even function, then $b_k = 0$ (this can also be gotten by calculations).

Then, the Fourier series is

$$(\mathcal{F}f)(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k+1} - \frac{1}{2k-1}.$$

3. Determine the Fourier series of the function $f(x) = \sin^2(x)\cos(2x)$.

   Solution: Instead of complicated calculations, we can use some identities: first, $\sin^2(x) = \frac{1-\cos(2x)}{2}$:

$$f(x) = \sin^2(x)\cos(x) = \frac{1-\cos(2x)}{2}\cos(2x) = \frac{1}{2}\cos(2x) - \frac{1}{2}\cos^2(2x) =$$

   Now, from the identities $\sin^2(2x) + \cos^2(2x) = 1$ and $\cos(4x) = \cos^2(2x) - \sin^2(2x)$ we get $\cos^2(2x) = \frac{1+\cos(4x)}{2}$, meaning that

$$f(x) = \frac{1}{2}\cos(2x) - \frac{1}{2}\frac{1+\cos(4x)}{2} = \frac{1}{2}\cos(2x) - \frac{1}{4} - \frac{1}{4}\cos(4x),$$

   which is a Fourier series with $a_0 = -\frac{1}{4}$, $a_2 = \frac{1}{2}$, $a_4 = -\frac{1}{4}$ and all the other coefficients are zero.

4. Determine the Fourier series of the following function:

$$f(x) = \begin{cases} -2 & \text{if } -2 \le x < -1, \\ x & \text{if } -1 \le x < 1, \\ 2 & \text{if } \quad 1 \le x < 2, \end{cases}$$

   and $f(x+4) = f(x)$.

   Solution: Here $P = 4$.

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x) = 0,$$

   since this is an odd function. Similarly, $a_k = 0$ since f is an odd function.

$$b_k = \frac{2}{4}\int_{-2}^{2} f(x)\sin\left(\frac{2\pi}{4}kx\right)dx =$$

$$= \frac{1}{2}\left(\int_{-2}^{-1}(-2)\sin\left(\frac{\pi k}{2}x\right)dx + \int_{-1}^{1} x\sin\left(\frac{\pi k}{2}x\right)dx + \int_{1}^{2} 2\sin\left(\frac{\pi k}{2}x\right)dx\right) =$$

we can combine the first and the last integrals:

$$= \frac{1}{2}\left(\int_{-1}^{1} x\sin\left(\frac{\pi k}{2}x\right)dx + 2\int_{1}^{2} 2\sin\left(\frac{\pi k}{2}x\right)dx\right) =$$

Now, since

$$\int x\sin\left(\frac{\pi k}{2}x\right)dx = -x\frac{\cos\left(\frac{k\pi}{2}x\right)}{\frac{k\pi}{2}} + \int \frac{\cos\left(\frac{k\pi}{2}x\right)}{\frac{k\pi}{2}}dx,$$

we get

$$= \frac{1}{2}\left(\left[-x\frac{\cos\left(\frac{k\pi}{2}x\right)}{\frac{k\pi}{2}}\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos\left(\frac{k\pi}{2}x\right)}{\frac{k\pi}{2}}dx - 4\left[\frac{\cos\left(\frac{k\pi}{2}x\right)}{\frac{k\pi}{2}}\right]_{1}^{2}\right) =$$

$$= \frac{1}{k\pi}\left(-\cos\left(\frac{k\pi}{2}\right) + \cos\left(-\frac{k\pi}{2}\right)\right) + \frac{1}{2}\left[\frac{\sin\left(\frac{k\pi}{2}x\right)}{\left(\frac{k\pi}{2}\right)^2}\right]_{-1}^{1} - \frac{4}{k\pi}\left[\cos(k\pi) - \cos\left(\frac{k\pi}{2}\right)\right] =$$

Now we use the fact that the cosine function is even, so the first term is zero:

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^2} - \frac{\sin\left(-\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^2}\right] - \frac{4}{k\pi}\left[\cos(k\pi) - \cos\left(\frac{k\pi}{2}\right)\right] =$$

$$= \frac{2}{k^2\pi^2}\left[\sin\left(\frac{k\pi}{2}\right) + \sin\left(\frac{k\pi}{2}\right)\right] - \frac{4}{k\pi}\left[\cos(k\pi) - \cos\left(\frac{k\pi}{2}\right)\right] =$$

$$= \frac{4}{k^2\pi^2}\sin\left(\frac{k\pi}{2}\right) - \frac{4}{k\pi}\left[\cos(k\pi) - \cos\left(\frac{k\pi}{2}\right)\right] =$$

Depending on the value of k, we have four cases:

- If $k = 4n$ (n is a positive integer), then $\cos(k\pi) = 1$, $\sin\left(\frac{k\pi}{2}\right) = 0$ and $\cos\left(\frac{k\pi}{2}\right) = 1$. In these cases the coefficient is zero.
- If $k = 4n+1$ (n is a positive integer), then $\cos(k\pi) = -1$, $\sin\left(\frac{k\pi}{2}\right) = 1$ and $\cos\left(\frac{k\pi}{2}\right) = 0$. In these cases the coefficient is $\frac{4}{k^2\pi^2} + \frac{4}{k\pi}$.
- If $k = 4n+2$ (n is a positive integer), then $\cos(k\pi) = 1$, $\sin\left(\frac{k\pi}{2}\right) = 0$ and $\cos\left(\frac{k\pi}{2}\right) = -1$. In these cases the coefficient is $-\frac{8}{k\pi}$.
- If $k = 4n+3$ (n is a positive integer), then $\cos(k\pi) = -1$, $\sin\left(\frac{k\pi}{2}\right) = -1$ and $\cos\left(\frac{k\pi}{2}\right) = 0$. In these cases the coefficient is $-\frac{4}{k^2\pi^2} + \frac{4}{k\pi}$.

5. Calculate the following numerical sum, if we know the result of Exercise 1.

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}.$$

Solution: From Exercise 1 we know that the Fourier series of that function was

$$(\mathcal{F}f)(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} -\frac{4}{\pi(2k-1)^2}\cos((2k-1)x) + \frac{4}{2k-1}\sin((2k-1)x).$$

If we substitute here $x = 0$, then

$$(\mathcal{F}f)(0) = -\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}.$$

Now we use the following theorem:

Theorem: Let f be a periodic function, which is continuous apart from finitely many points where it can have jumps or removable discontinuities. Then,

$$(\mathcal{F}f)(x_0) = \frac{\lim_{x\to x_0+0} f(x) + \lim_{x\to x_0-0} f(x)}{2}.$$

In this case if $x_0 = 0$, then we know that $\lim_{x\to x_0+0} f(x) = 0$ and $\lim_{x\to x_0-0} f(x) = (-2\pi)$, so the equation we have is

$$-\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{0+(-2\pi)}{2}.$$

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

6. Calculate the Fourier sine series of the function $f(x) = \cos(x)$, $x \in (0, \pi)$.

   Solution: We get a Fourier sine function (i.e. $a_k = 0$ for every $k \in \mathbb{Z}^+$) if the function is odd. For this, let us extend the function to the interval $(-\pi, \pi)$ according to the following rule:

$$f(x) = \begin{cases} -\cos(x) & \text{if } -\pi \le x < 0, \\ \cos(x) & \text{if } \quad 0 \le x < \pi, \end{cases}$$

   We can also make it a periodic function by the rule $f(x + 2\pi) = f(x)$, and then f is defined for every $x \in \mathbb{R}$.

   Since this is odd, we get $a_k = 0$ for $k > 0$, and also $a_0 = 0$ since the integral of f is zero on $(-\pi, \pi)$ (this can be seen either by the symmetry of the function, or by calculations.)

   For b_k, we get

$$b_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(x)\sin(kx)dx + \int_{0}^{\pi} \cos(x)\sin(kx)dx\right) =$$

   Now we use the same trick as in Exercise 2., namely

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) - \sin(\beta-\alpha).$$

   Then,

$$= \frac{1}{\pi}\left(-\int_{-\pi}^{0} \sin((k+1)x) - \sin((1-k)x)dx + \int_{0}^{\pi} \sin((k+1)x) - \sin((1-k)x)dx\right) =$$

$$= \frac{1}{\pi}\left(-\left[-\frac{\cos((k+1)x)}{k+1}\right]_{-\pi}^{0} + \left[-\frac{\cos((1-k)x)}{1-k}\right]_{-\pi}^{0} + \left[-\frac{\cos((k+1)x)}{k+1}\right]_{0}^{\pi} - \left[-\frac{\cos((1-k)x)}{1-k}\right]_{0}^{\pi}\right) =$$

$$= \frac{1}{\pi}\left(\left[\frac{1}{k+1} - \frac{\cos((k+1)(-\pi))}{k+1}\right] - \left[\frac{1}{1-k} - \frac{\cos((1-k)(-\pi))}{1-k}\right] - \right.$$

$$\left. - \left[\frac{\cos((k+1)\pi)}{k+1} - \frac{1}{k+1}\right] + \left[\frac{\cos((1-k)\pi)}{1-k} - \frac{1}{1-k}\right]\right) =$$

$$= \frac{1}{\pi}\left(\frac{2}{k+1} - \frac{2}{1-k} - \frac{2\cos((k+1)\pi)}{k+1} + \frac{2\cos((1-k)\pi)}{1-k}\right)$$

   Now we have two cases:

   - If k is odd, then $b_k = 0$.
   - If k is even, then $b_k = \dfrac{1}{\pi}\left(\dfrac{4}{k+1} - \dfrac{4}{1-k}\right) = \dfrac{-8k}{\pi(1-k^2)}$.

7. How should we extend the following function to get a Fourier sine series?

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1, \\ -\frac{2}{3}x + \frac{8}{3} & \text{if } 1 \leq x < 4. \end{cases}$$

Solution: By extending the function onto $[-4, 4]$ as

$$f(x) = \begin{cases} -\frac{2}{3}x - \frac{8}{3} & \text{if } 1 \leq x < 4, \\ 2x & \text{if } -1 \leq x < 1, \\ -\frac{2}{3}x + \frac{8}{3} & \text{if } -4 \leq x < -1, \end{cases}$$

we get an odd function, resulting in a sine series.

8. Calculate the Fourier cosine series of the function $f(x) = \sin(x)$, $x \in (0, \pi)$.

Solution: We get a Fourier cosine function (i.e. $b_k = 0$ for every k) if the function is even. For this, let us extend the function to the interval $(-\pi, \pi)$ according to the following rule:

$$f(x) = \begin{cases} -\sin(x) & \text{if } -\pi \leq x < 0, \\ \cos(x) & \text{if } \quad 0 \leq x < \pi, \end{cases}$$

We can also make it a periodic function by the rule $f(x + 2\pi) = f(x)$, and then f is defined for every $x \in \mathbb{R}$.

Note that this function is the same as $|\sin(x)|$, whose Fourier series was calculated in Exercise 2, so we can use the result of that one.