Math G2 Practices 9

Multivariable functions I.: limits, continuity, partial differentiation

1 Limits of multivariable functions, continuity

Theorem: The following two are equaivalent:

- $\lim\limits_{x \to a} f(x) = A$,
- For every sequence $\lim_{k\to\infty} x_k = a$, the limit $\lim_{x_k \to a} f(x_k)$ exists and it is A.

Theorem: The following two are equaivalent:

- $\lim\limits_{x \to a} f(x) = f(a)$,
- f is continuous at point a.

Theorem: If $\lim\limits_{x \to a} f(x)$ exists, then

- the order of the two limits can be switched,
- the limit $\lim_{x_k \to a} f(x_k)$ should be the same for any sequence x_k converging to a (even if the points are on a straight line or on some other curve).

1. Calculate the following limit!

$$\lim_{x\to 2, y\to\infty} \frac{2xy-1}{y+1}$$

Solution:

$$\lim_{x\to 2, y\to\infty} \frac{2xy-1}{y+1} = \lim_{x\to 2}\lim_{y\to\infty} \frac{2xy-1}{y+1} = \lim_{x\to 2}\lim_{y\to\infty} \frac{2x-\frac{1}{y}}{1+\frac{1}{y}} = \lim_{x\to 2} \frac{2x-0}{1+0} = 4$$

Remark: If we would like to be more precise, the above calculation can be done by the formal definition too.

2. Calculate the following limit!

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2x^2+y^2}$$

Solution:

$$\lim_{x\to 0}\lim_{y\to 0} \frac{x^2+y^2}{2x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

However, if we consider the other order of the limits, we get

$$\lim_{y\to 0}\lim_{x\to 0} \frac{x^2+y^2}{2x^2+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = \lim_{y\to 0} 1 = 1$$

Since these two values are different, there is no limit at $(0,0)$.

3. Calculate the following limit!

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

Solution: Since both order of the limits results in a limit of form $\frac{0}{0}$, we should apply some other technique.

We will transform the problem into polar coordinates: namely, let us use the formulas

$$x = r\cos(\varphi), \qquad y = r\sin(\varphi),$$

and if $(x, y) \to (0, 0)$, in polar coordinates it means that $r \to 0$.

After substitution:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos(\varphi)\sin^2(\varphi)}{r^2\cos^2(\varphi)+r^2\sin^2(\varphi)} = \lim_{r\to 0} \frac{r^3\cos(\varphi)\sin^2(\varphi)}{r^2(\cos^2(\varphi)+\sin^2(\varphi))} =$$

Now we use the fact that $\cos^2(\varphi) + \sin^2(\varphi) = 1$, meaning that

$$= \lim_{r\to 0} \frac{r^3\cos(\varphi)\sin^2(\varphi)}{r^2} = \lim_{r\to 0} r\cos(\varphi)\sin^2(\varphi) = 0,$$

in the last step, we used the fact that if a bounded term is multiplied by a term tending to zero, then the product tends to zero.

4. Calculate the following limit!

$$\lim_{(x,y)\to(0,3)} \frac{\sqrt{x^2+y^2-6y+10}-1}{x^2+y^2-6y+9}$$

Solution: Since in this case (x, y) tends to $(0, 3)$, let us use the polar coordinates around the point $(0, 3)$: namely,

$$x = r\cos(\varphi), \qquad y = 3 + r\sin(\varphi),$$

and if $(x, y) \to (0, 3)$, in polar coordinates it means that $r \to 0$.

After substitution:

$$\lim_{(x,y)\to(0,3)} \frac{\sqrt{x^2+y^2-6y+10}-1}{x^2+y^2-6y+9} = \lim_{(x,y)\to(0,3)} \frac{\sqrt{x^2+(y-3)^2+1}-1}{x^2+(y-3)^2} =$$

$$= \lim_{r\to 0} \frac{\sqrt{r^2\cos^2(\varphi)+r^2\sin^2(\varphi)+1}-1}{r^2\cos^2(\varphi)+r^2\sin^2(\varphi)} =$$

Now we use the fact that $\cos^2(\varphi) + \sin^2(\varphi) = 1$, meaning that

$$= \lim_{r\to 0} \frac{\sqrt{r^2+1}-1}{r^2} =$$

Now we use the l'Hospital rule, i.e. differentiating the numerator and the denominator:

$$= \lim_{r\to 0} \frac{\frac{1}{\sqrt{r^2+1}}\frac{1}{2}2r}{2r} = \frac{1}{2}.$$

5. Calculate the following limit!

$$\lim_{x\to\infty, y\to\infty} \frac{x+y}{x^2-xy+y^2}$$

Solution: We are going to use polar coordinates. Since $x \to \infty, y \to \infty$, the center of the polar coordinates is not that important, we can use

$$x = r\cos(\varphi), \qquad y = r\sin(\varphi),$$

with $r \to \infty$.

After substitution:

$$\lim_{x\to\infty, y\to\infty} \frac{x+y}{x^2-xy+y^2} = \lim_{r\to\infty} \frac{r\cos(\varphi)+r\sin(\varphi)}{r^2\cos^2(\varphi)-r^2\sin(\varphi)\cos(\varphi)+r^2\sin^2(\varphi)} =$$

$$= \lim_{r\to\infty} \frac{r(\cos(\varphi)+\sin(\varphi))}{r^2(\cos^2(\varphi)+\sin^2(\varphi))-r^2\sin(\varphi)\cos(\varphi)} = \lim_{r\to\infty} \frac{r(\cos(\varphi)+\sin(\varphi))}{r^2(1-\sin(\varphi)\cos(\varphi))} =$$

$$= \lim_{r\to\infty} \frac{1}{r}\frac{\cos(\varphi)+\sin(\varphi)}{1-\sin(\varphi)\cos(\varphi)} = 0,$$

where we used the fact that the first term goes to zero and the second one is bounded.

6. Calculate the following limit!

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{x^2y^2}$$

Solution: The main idea of this exercise is that if

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{x^2y^2} = A,$$

then

$$\ln(A) = \lim_{(x,y)\to(0,0)} \ln\left(\left(x^2+y^2\right)^{x^2y^2}\right) = \lim_{(x,y)\to(0,0)} (x^2y^2)\ln\left((x^2+y^2)\right) =$$

Now let us use polar coordinates:

$$= \lim_{r\to 0}(r^4\cos^2(\varphi)\sin^2(\varphi))\ln(r^2\cos^2(\varphi)+r^2\sin^2(\varphi)) = \lim_{r\to 0}(r^4\cos^2(\varphi)\sin^2(\varphi))\ln(r^2) =$$

$$= \lim_{r\to 0} r^4\ln(r^2)(\cos^2(\varphi)\sin^2(\varphi)) =$$

Now we use the fact that (we use the rule of l'Hospital)

$$\lim_{r\to 0} r^4\ln(r^2) = \lim_{r\to 0}\frac{\ln(r^2)}{r^{-4}} = \lim_{r\to 0}\frac{\frac{1}{r^2}2r}{(-4)r^{-5}} = \lim_{r\to 0}\frac{\frac{2}{r}}{(-4)r^{-5}} = \lim_{r\to 0}\frac{r^4}{-2} = 0.$$

Then, the previous limit is zero, since we have a term that tends to zero and the second one is bounded. It means that $\ln(A) = 0$, so $A = 1$.

7. How should we change the value of parameter c such that the following function is continuous?

$$f(x) = \begin{cases} \dfrac{\sqrt{x^2+y^2-6y+10}-1}{x^2+y^2-6y+9} & \text{if } (x,y)\neq(0,3), \\ c & \text{if } (x,y)=(0,3). \end{cases}$$

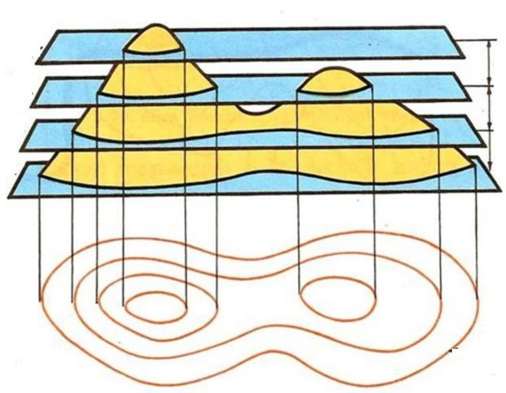
Solution: By Exercise 4.,

$$\lim_{(x,y)\to(0,3)} \frac{\sqrt{x^2+y^2-6y+10}-1}{x^2+y^2-6y+9} = \frac{1}{2},$$

so the function is continuous if $c = \dfrac{1}{2}$.

2 Partial derivatives

The derivative meant the rate of change in the case of one-variable functions, which somehow means the rate the function increases or decreases. In the two-variable case, we have a problem: consider for example the "valley" between these two "hills" on the image below: if we start to move towards one of the peaks of these hills, the function will increase; however, if we move to the other directions (down the hills), the function might decrease. So it is important to tell the direction in which we would like to compute this rate of change, since it might be considerably different when we consider these derivatives in different directions. The most commonly used directions are the directions of the two axis (the direction of the x and the y-axis), called **partial derivatives**.



When we are calculating the partial derivative with respect to some variable x_k, we consider all the other variables to be constants, and proceed further as in the one-variable case.

8. Calculate the partial derivatives of the following function!

$$f(x,y) = e^{x^2y} - 2x^2y^3\sin(x+y)$$

Solution: To calculate the partial derivative with respect to x, let us consider y to be a constant:

$$\frac{\partial f}{\partial x} = e^{x^2y}2xy - (4xy^3\sin(x+y) + 2x^2y^3\cos(x+y)(1+0))$$

To calculate the partial derivative with respect to y, let us consider x to be a constant:

$$\frac{\partial f}{\partial y} = e^{x^2y}x^2 - (6x^2y^2\sin(x+y) + 2x^2y^3\cos(x+y)(0+1))$$

9. Calculate the partial derivatives of the following function!

$$f(x,y,z) = xy\sin(z) + zx\ln(y) + ye^x$$

Solution: To calculate the partial derivative with respect to x, let us consider y and z to be a constants:

$$\frac{\partial f}{\partial x} = y\sin(z) + z\ln(y) + ye^x$$

To calculate the partial derivative with respect to y, let us consider x and z to be a constants:

$$\frac{\partial f}{\partial y} = x\sin(z) + zx\frac{1}{y} + e^x$$

To calculate the partial derivative with respect to z, let us consider x and y to be a constants:

$$\frac{\partial f}{\partial z} = xy\cos(z) + x\ln(y)$$

10. Calculate the partial derivatives of the following implicit function!

$$x\cos(y) + y\cos(z(x,y)) + z(x,y)\cos(x) = 1$$

Solution: To calculate the partial derivative with respect to x, let us take the partial derivative of the equation with respect to x:

$$\cos(y) + y(-\sin(z(x,y))) \cdot \frac{\partial z(x,y)}{\partial x} + \frac{\partial z(x,y)}{\partial x}\cos(x) + z(x,y)(-\sin(x)) = 0.$$

Then, by some regrouping of the terms:

$$\frac{\partial z(x,y)}{\partial x} = \frac{z(x,y)\sin(x) - \cos(y)}{\cos(x) - y\sin(z(x,y))}.$$

To calculate the partial derivative with respect to y, let us take the partial derivative of the equation with respect to y:

$$-x\sin(y) + \cos(z(x,y)) - y\sin(z(x,y))\frac{\partial z(x,y)}{\partial y} + \frac{\partial z(x,y)}{\partial y}\cos(x) = 0.$$

Then, by some regrouping of the terms:

$$\frac{\partial z(x,y)}{\partial y} = \frac{x\sin(y) - \cos(z(x,y))}{\cos(x) - y\sin(z(x,y))}.$$

11. Calculate the derivative of the following composition of functions!

$$f(x) = g(u(x), v(x)) = \arctan(u(x)) + v(x),$$

where $u(x) = e^{2x}$ and $v(x) = \cos(2x)$.

Solution: We can solve such exercises in two different ways.

First method: Let us use the chain rule, which states that

$$\frac{df(x)}{dx} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{du(x)}{dx} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{dv(x)}{dx}$$

Then, in our case:

$$\frac{df(x)}{dx} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{du(x)}{dx} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{dv(x)}{dx} = \frac{1}{1+u^2(x)} 2e^{2x} + 1 \cdot (-2\sin(2x)) =$$

$$= \frac{1}{1+(e^{2x})^2} 2e^{2x} + 1 \cdot (-2\sin(2x))$$

Second method: Let us calculate $f(x)$!

$$f(x) = \arctan(u(x)) + v(x) = \arctan(e^{2x}) + \cos(2x).$$

This is a one-variable function, so we can calculate its derivative by the rules learned in Math G1:

$$\frac{df(x)}{dx} = \frac{1}{(e^{2x})^2+1} e^{2x} 2 + 2(-\sin(2x)).$$

Note that we got the same result in both cases.

12. Calculate the derivative of the following composition of functions!

$$f(x) = g(u(x), v(x)) = \ln(u(x) \cdot v(x)),$$

where $u(x) = \tan(x)$ and $v(x) = \sqrt{x}$.

Solution: We can solve such exercises in two different ways.

First method: Let us use the chain rule, which states that

$$\frac{df(x)}{dx} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{du(x)}{dx} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{dv(x)}{dx}$$

Then, in our case:

$$\frac{df(x)}{dx} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{du(x)}{dx} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{dv(x)}{dx} = \frac{1}{u(x) \cdot v(x)} v(x) \frac{1}{\cos^2(x)} + \frac{1}{u(x) \cdot v(x)} u(x) \frac{1}{2\sqrt{x}} =$$

$$= \frac{1}{\tan(x) \cdot \sqrt{x}} \sqrt{x} \frac{1}{\cos^2(x)} + \frac{1}{\tan(x) \cdot \sqrt{x}} \tan(x)) \frac{1}{2\sqrt{x}}$$

Second method: Let us calculate $f(x)$!

$$f(x) = \ln(u(x) \cdot v(x)) = \ln(\tan(x)\sqrt{x}).$$

This is a one-variable function, so we can calculate its derivative by the rules learned in Math G1:

$$\frac{df(x)}{dx} = \frac{1}{\tan(x)\sqrt{x}} \left(\frac{1}{\cos^2(x)} \sqrt{x} + \tan(x) \frac{1}{2\sqrt{x}} \right).$$

Note that we got the same result in both cases.

13. Calculate the partial derivatives of the following composition of functions!

$$f(x,y) = g(u(x,y), v(x,y)) = \arcsin(u(x) \cdot v(x)),$$

where $u(x,y) = e^{xy}$ and $v(x,y) = 2x - 2xy$.

Solution: We can solve such exercises in two different ways.

First method: Let us use the chain rule, which states that

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{\partial u(x,y)}{\partial x} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{\partial v(x,y)}{\partial x}$$

Then, in our case:

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{\partial u(x,y)}{\partial x} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{\partial v(x,y)}{\partial x} =$$

$$= \frac{1}{\sqrt{1-(u(x,y)v(x,y))^2}} v(x,y) y e^{xy} + \frac{1}{\sqrt{1-(u(x,y)v(x,y))^2}} u(x,y)(2-2y) =$$

$$= \frac{1}{\sqrt{1-(e^{xy}(2x-2xy))^2}} (2x-2xy) y e^{xy} + \frac{1}{\sqrt{1-(e^{xy}(2x-2xy)^2}} e^{xy}(2-2y)$$

The other partial derivative can be calculated similarly:

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial g(u,v)}{\partial u} \cdot \frac{\partial u(x,y)}{\partial y} + \frac{\partial g(u,v)}{\partial v} \cdot \frac{\partial v(x,y)}{\partial y} =$$

$$= \frac{1}{\sqrt{1-(u(x,y)v(x,y))^2}} v(x,y) x e^{xy} + \frac{1}{\sqrt{1-(u(x,y)v(x,y))^2}} u(x,y)(-2x) =$$

$$= \frac{1}{\sqrt{1-(e^{xy}(2x-2xy)^2}} (2x-2xy) x e^{xy} + \frac{1}{\sqrt{1-(e^{xy}(2x-2xy)^2}} e^{xy}(-2x)$$

Second method: Let us calculate $f(x,y)$!

$$f(x,y) = \arcsin(u(x,y) \cdot v(x,y)) = \arcsin(e^{xy} \cdot (2x-2xy)).$$

Then, the partial derivatives can be calculated directly:

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{\sqrt{1-(e^{xy}(2x-2xy))^2}} \left(e^{xy} y(2x-2xy) + e^{xy}(2-2y)\right)$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{\sqrt{1-(e^{xy}(2x-2xy))^2}} \left(e^{xy} x(2x-2xy) + e^{xy}(-2x)\right)$$

Note that we got the same result in both cases.