Mock exam

Partial differential equations

2022

Exam grade:

- $40 - 59$ 2,
- $60 - 79$ 3,
- $80 - 99$ 4,
- 5 above 100.

1 First part: definitions, theorems

For every correct answer you'll get 9 points.

1. Let us consider the equation

$$\sum_{j,k=1}^{n} a_{j,k} \partial_j \partial_k u + \sum_{j=1}^{k} b_j \partial_j u + cu = f.$$

   In which case is this equation called elliptic?

2. Let us consider the interval $[0,1] \subset \mathbb{R}$ and another one Ω for which $[0,1] \subset \Omega$. Is it possible to construct such functions φ_1, φ_2 and φ_3, for which $\varphi_1, \varphi_2, \varphi_3 \in C_0^\infty(\Omega)$, and also $0 \le \varphi_i \le 1$ $(i = 1,2,3)$ and $\varphi_1(x) + \varphi_2(x) + \varphi_3(x) = 1$ $(\forall x \in [0,1])$? If yes, give the theorem that guarantees this property, if not, give a counterexample!

3. Define the convergence $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$! What do we call distributions in the theory of PDEs?

4. Define the Cartesian product of distributions $u(\phi) \in \mathcal{D}'(\mathbb{R}^n)$ and $v(\psi) \in \mathcal{D}'(\mathbb{R}^m)$!

5. Assume that $u, v \in \mathcal{D}'(\Omega)$ and $u * v$ exists. Is it possible that $\partial^\alpha(u * v) \neq u * (\partial^\alpha v)$?

6. State the maximum principle for the heat equation defined on $(t, x) \in [0, T] \times \mathbb{R}$.

7. Let us consider operator $L : \mathrm{Dom}(L) \subset L^2(\Omega) \to L^2(\Omega)$ with domain

$$\mathrm{Dom}(L) := \left\{ u \in C^2(\Omega \cap C^1(\overline{\Omega})) : Lu \in L^2(\Omega), gu|_{\partial\Omega} + h\partial_\mu u|_{\partial\Omega} = 0 \right\}$$

   defined as

$$Lu = -\mathrm{div}(\mathrm{grad}(u))$$

   in which $p \in C^1(\overline{\Omega})$, $p(x) > 0$, $q \in C(\overline{\Omega})$, $g, h \in C(\partial\Omega)$. What can we say about operator L and its eigenvalues and eigenfunctions?

8. Define the Sobolev space $H^k(\Omega)$!

9. Let us assume that the first boundary value problem has a weak (or general) solution. What do we have to prove to show that it is also a classical solution?

2 Part two: theorems and proofs

Topic: wave equation

1. Define the classical solution of the wave equation! (5 points)
2. Extend functions u and f onto $\mathbb{R}^n$. (5 points)
3. State the result about the 'wave equation' for $T_{\tilde{u}}$. (5 points)
4. Prove the previous statement. (45 points)
   Hint: Compute the left-hand side, and then observe the three terms separately.
5. Define the general (or weak) form of the wave equation. (5 points)
6. State the connection between the classical and the weak solution. (5 points)
7. State the result about the uniqueness of the solution of the general equation. (5 points)
8. What can we say about the number of solutions of the classical problem? Why? (5 points)