Practice 1

Simple equations

I. **Simple equations.**

1. We seek the $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (classical) solutions of the following equations:

   (a) $\partial_y u = 0$
   Solution: By integrating with respect to variable y: $u(x, y) = f(x)$, where $f(x) \in C^1(\mathbb{R})$.

   (b) $\partial_{xy} u = 0$
   Solution: Or equation is
$$\partial_{xy} u = \partial_x \left(\partial_y u(x, y)\right) = 0.$$
   By integrating with respect to variable x:
$$\partial_y u(x, y) = f(y).$$
   Then by integrating with respect to y:
$$u(x, y) = F(y) + G(x),$$
   in which $f,\ F,\ G \in C^2(\mathbb{R})$.

   (c) $\partial_{xy} u = \dfrac{4xy}{(x^2 + y^2)^2}$
   Solution: By integrating with respect to x:
$$\partial_y u = -\frac{2y}{x^2 + y^2} + f(y).$$
   Then by integrating with respect to y:
$$u(x, y) = -\ln(x^2 + y^2) + F(y) + G(x),$$
   in which $f, F, G \in C^2(\mathbb{R})$.
   Remark: This solution is only defined on $\mathbb{R}^2 \backslash \{(0{,}0)\}$.

   (d) $\partial_{xy} u + 2x\partial_y u = x$
   Solution: Let us define the following new function $v : \mathbb{R} \to \mathbb{R}$: $v(x; y) = \partial_y u(x, y)$ (here we suppose that y is only a parameter in v). Consequently, we get the following equation:
$$v'(x; y) + 2xv(x; y) = x. \tag{1}$$
   (Here $'$ is a derivation in x, and the notation $(x; y)$ means that we think of this function as a function in x, and y is only a parameter in it.)
   First method: Let us multiply both sides by e^{x^2}:
$$e^{x^2} v'(x; y) + e^{x^2} 2xv(x; y) = e^{x^2} x.$$
   Note that on the left side, we have $(e^{x^2} v(x, y))'$, and on the right hand side $\left(\frac{1}{2} e^{x^2}\right)'$, so we can integrate both sides and we get
$$e^{x^2} v(x, y) = \frac{1}{2} e^{x^2} + f(y).$$
   (The term $f(y)$ appears since y is a parameter in v.) If we multiply both sides by e^{-x^2}:
$$v(x, y) = \frac{1}{2} + e^{-x^2} f(y).$$
   Since $v(x; y) = \partial_y u(x, y)$, we get that
$$u(x, y) = \frac{1}{2} y + F(y) e^{-x^2} + G(x),$$
   where $f, F, G \in C^2(\mathbb{R})$.
   Second method: We can solve equation (1) by searching for a particular solution, and add it to the homogeneous form of equation (1) - see your ODE practice notes for details.
   Third method: We can solve the problem without introducing the function $v(x; y)$: let us multiply our initial problem by e^{x^2}, so we have $e^{x^2} \partial_{xy} u + 2xe^{x^2} \partial_y u$ on the left, which is $\partial_{xy} \left(e^{x^2} u(x, y)\right)$.
   Fourth method: We can even integrate the initial problem in y, and we then get an ODE, which can be solved similarly as the one in the first method.

(e) $\partial_x u = \partial_y u$

Solution:

First method: Let us define vector v as $v = (1, -1)^T$. Then our initial problem can be expressed as

$$0 = (\partial_x u, \partial_y u) \cdot v = \partial_v u.$$

This means that the derivative of u in direction v is zero, meaning that our function u is constant on the $x+y = c,\ c \in \mathbb{R}$ lines. This means that $u(x, y) = f(x+y)$, where $f \in C^1(\mathbb{R})$.

Second method: Let us use the following transformation of variables: $\xi = x + y$, $\eta = x - y$, and define a new function: $u(x, y) = U(\xi, \eta)$. Then:

$$(\partial_x u, \partial_y u) = u'(x, y) = U'(\xi, \eta) \begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{pmatrix} = (\partial_\xi U, \partial_\eta U) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This means that

$$\begin{cases} \partial_x u = \partial_\xi U + \partial_\eta U, \\ \partial_y u = \partial_\xi U - \partial_\eta U. \end{cases}$$

Then by substituting these into the original equation we get that

$$0 = \partial_x u - \partial_y u = 2\partial_\eta U,$$

from which we get that $\partial_\eta U = 0$, which means that $U(\xi, \eta) = f(\xi)$, so the solution of our original problem is $u(x, y) = f(x + y)$ where $f \in C^1(\mathbb{R})$.

(f) $\partial_x^2 u - \partial_y^2 u = 0$

Solution: If we apply the same method as in the second method of problem (e), we have $U(\xi, \eta) = u(x, y)$, and also $\xi = x + y$ and $\eta = x - y$. We have also computed that

$$\partial_x u = \partial_\xi U + \partial_\eta U,$$

$$\partial_y u = \partial_\xi U - \partial_\eta U.$$

Therefore,

$$\partial_x(\partial_x u) = \partial_\xi(\partial_\xi U + \partial_\eta U) + \partial_\eta(\partial_\xi U + \partial_\eta U) = \partial_{\xi\xi} U + \partial_{\xi\eta} U + \partial_{\eta\xi} U + \partial_{\eta\eta} U,$$

and also

$$\partial_{yy} u = \partial_\xi(\partial_\xi U - \partial_\eta U) - \partial_\eta(\partial_\xi U - \partial_\eta U) = \partial_{\xi\xi} U - \partial_{\xi\eta} U - \partial_{\eta\xi} U + \partial_{\eta\eta} U.$$

By subtracting these from each other we get $4\partial_{\xi\eta} U = 0$, and according to problem (b), the solution is $u = F(x - y) + G(x + y)$ ($F, G \in C^2(\mathbb{R})$).

(g) $\partial_{xx} u - a^2 \partial_{yy} u = 0$

Solution: At first, let us suppose that $a \neq 0$. Then let us define a function $v(x, y) : \mathbb{R}^2 \to \mathbb{R}$ as $v(x, y) := u(x, ay)$. Then

$$\partial_{yy} v(x, y) = a^2 \partial_{yy} u(x, ay)$$

and

$$\partial_{xx} v(x, y) = \partial_{xx} u(x, ay)$$

so because of our initial problem we have

$$\partial_{xx} v(x, y) - \partial_{yy} v(x, y) = 0.$$

However, because of (f), we get that

$$v(x, y) = G(x + y) + F(x - y).$$

$$u(x, y) = v\left(x, \frac{y}{a}\right) = G\left(x + \frac{y}{a}\right) + F\left(x - \frac{y}{a}\right) = G\left(\frac{1}{a}(ax + y)\right) + F\left(\frac{1}{a}(ax - y)\right) =$$
$$= g(ax + y) + f(-ax + y),$$

in which $f, g, F, G \in C^2(\mathbb{R})$.

Now let us observe the case $a = 0$. Then our equation simplifies to

$$\partial_x^2 u(x, y) = 0.$$

From which we get (through two integrations with respect to x):

$$u(x, y) = xf(y) + g(y),$$

in which $f, g \in C^2(\mathbb{R})$.

Remark: The one dimensional wave equation has the form

$$\partial_t^2 u - a^2 \partial_x^2 u = 0$$

Then, according to (g), its solution has the form

$$u(t,x) = f(x+at) + g(x-at)$$

in which the left hand term on the right hand side corresponds to a wave travelling to the left with speed a, while the right-hand term corresponds to a wave travelling to the right.

2. Give the solution $u \in C^2(\mathbb{R}^3)$ to the following equation:

$$\partial_x^2 u(x,y,z) = 0$$

Solution: By simple integrations:

$$\partial_x \left(\partial_x u(x,y,z)\right) = 0$$

$$\partial_x u(x,y,z) = f(y,z)$$

$$u(x,y,z) = xf(y,z) + g(y,z)$$

in which $f, g \in C^2(\mathbb{R}^2)$.

3. Give the solution $u \in C^2(\mathbb{R}^2)$ of the following equation!

(a)

$$\begin{cases} \partial_{xy} u = x + y, \\ u(x,x) = x, \\ \partial_x u(x,x) = 0. \end{cases} \tag{2}$$

Solution: Since $\partial_{xy} u = x + y$, then by integration

$$\partial_x u(x,y) = xy + \frac{1}{2}y^2 + f(x)$$

Then by the third line of (2), we have

$$0 = \partial_x u(x,x) = \frac{3}{2}x^2 + f(x)$$

so $f(x) = -\frac{3}{2}x^2$, which also means that

$$\partial_x u(x,y) = xy + \frac{1}{2}y^2 - \frac{3}{2}x^2$$

$$u(x,y) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 - \frac{1}{2}x^3 + g(y)$$

Now we use the second line of (2):

$$x = u(x,x) = \frac{1}{2}x^3 + g(x)$$

from which we get that $g(x) = x - \frac{1}{2}x^3$, and our solution is

$$u(x,y) = \frac{1}{2}x^2 y + \frac{1}{2}xy^2 - \frac{1}{2}x^3 + y - \frac{1}{2}y^3$$

(b)

$$\begin{cases} \partial_x^2 u - \partial_y^2 u = 0, \\ u(0,y) = 1, \\ \partial_x u(0,y) = 1. \end{cases} \tag{3}$$

Solution: Because of problem 1. (f), the solution is in the form

$$u(x,y) = G(x+y) + F(x-y)$$

Then we use the boundary conditions:

$$u(0,y) = G(y) + F(-y) = 1 \tag{4}$$

$$\partial_x u(0,y) = G'(y) + F'(-y) = 1 \tag{5}$$

Now upon differentiating (4), we get

$$G'(y) - F'(-y) = 0$$

and if we add this up with (5):

$$2G'(y) = 1$$

$$G(y) = \frac{1}{2}y + c \qquad \Rightarrow \qquad F(-y) = 1 - G(y) = 1 - \frac{1}{2}y - c \qquad \Rightarrow \qquad F(y) = 1 + \frac{1}{2}y - c$$

From these, our solution is

$$u(x, y) = \frac{1}{2}(x + y) + c + 1 + \frac{1}{2}(x - y) - c = x + 1.$$

4. Search for the solutions of the following equation in the form $u(x, y) = X(x)Y(y)$:

$$\partial_x^2 u - \partial_y u = 0.$$

Solution: Since $u(x, y) = X(x)Y(y)$, then

$$\partial_{xx} u = X''(x)Y(y),$$

and

$$\partial_y u = X(x)Y'(y).$$

Since these previous two terms are equal, we have

$$X''(x)Y(y) = X(x)Y'(y),$$

$$\frac{X''(x)}{X(x)} = \frac{Y'(y)}{Y(y)}.$$

Since we have a one-variable function on both sides but with different variables, then the equality can only hold if these two are constant functions - let us denote their value by α. From this, we get the following equations:

$$X''(x) - \alpha X(x) = 0$$
$$Y'(y) - \alpha Y(y) = 0$$

The solution of the second one is $Y(y) = c_0\, e^{\alpha y}$ $(c \in \mathbb{R})$, and the solution of the first depends on the sign of α:

$$X(x) = \begin{cases} c_1 \sin(\sqrt{-\alpha}x) + c_2 \cos(\sqrt{-\alpha}x) & \text{if } \alpha < 0 \\ c_1 e^{\sqrt{\alpha}x} + c_2 e^{-\sqrt{\alpha}x} & \text{if } \alpha > 0 \\ c_1 x + c_2 & \text{if } \alpha = 0 \end{cases}$$

Then the solution is the product of the corresponding $X(x)$ and $Y(y)$.
Remark: It is worth mentioning that these are not all solutions of the problem, e.g. $x^2 + 2y$ is also a solution. However, the (possibly infinite) linear combination of the functions calculated above produces all solutions.

5. Give all $u : \mathbb{R}^2 \to \mathbb{R}$ polynomials for which

$$\Delta u = 0. \tag{6}$$

Solution: We can easily find such functions, e.g. x, y or $x^2 + y^2$ are fine. Now the question is: how to determine all such functions?
We use a theorem from complex analysis, which states that the solution of (6) has the form

$$u(x, y) = \operatorname{Re}\left[f(x + iy)\right]$$

where $f : \mathbb{C} \to \mathbb{C}$ is a regular (or holomorphic) function.
Now let us define our function as $f(z) := z^n$, or in other words $f(x + iy) = (x + iy)^n$. Here f is regular, so (6) holds for u if it is defined using this f. Also, $\operatorname{Re}(f)$ is a two-variable polynomial (since e.g. for $n = 1$, we have x, for $n = 2$ we have $x^2 - y^2$, for $n = 3$ we have $x^3 - 3xy^2$ and so on). But $\operatorname{Im}(f)$ is also a two-variable polynomial. This means that for all n, we have two, nth degree polynomials for which (6) holds. Our goal from now is to show that these polynomials are also the base of the vector space of nth degree polynomials. For this we need to prove two things: that they are linearly independent, and also that the vector space of the nth degree such polynomials has dimension of 2.
The linear independence is true, because of the different monomials inside the different functions.

For the dimension of the vector space, let us define

$$P_n := \{x^j y^k : j+k=n, \quad j,k \geq 0\}.$$

Then it is clear that the Laplace operator Δ maps P_n to P_{n-2}. Also, this map is surjective, since all elements of P_{n-2} are the image of one element in P_n (this can be proved by induction). However, surjectivity means that

$$\dim(\mathrm{Ker}(\Delta)) = \dim(P_n) - \dim(P_{n-2}),$$

(since there are no elements of P_n which are mapped somewhere else.)
We also know that $\dim(P_n) = n+1$, since if $j+k=n$ and k is determined, then for j we have $n+1$ different choices. Therefore,

$$\dim(\mathrm{Ker}(\Delta)) = \dim(P_n) - \dim(P_{n-2}) = n+1-(n-2+1) = 2$$

which is in fact the space of functions which are solutions of (6), so it truly has dimension 2. Consequently, we proved that this space has dimension 2, and we also gave two linearly independent elements, which means that it forms a base.

Thus, all polynomial solutions of (6) are linear combinations of $\mathrm{Re}((x+iy)^n)$ and $\mathrm{Im}((x+iy)^n)$.

Remark: Those functions for which (6) holds are called *harmonic functions*.

6. Suppose that for some $u \in C^4(\mathbb{R}^2)$, we have $\Delta u = 0$. Prove that if $v(x,y) := (x^2+y^2)u(x,y)$, then $\Delta^2 v = 0$.

Solution:

$$\partial_x v(x,y) = 2xu(x,y) + (x^2+y^2)\partial_x u(x,y)$$
$$\partial_y v(x,y) = 2yu(x,y) + (x^2+y^2)\partial_y u(x,y)$$

and also,

$$\partial_x^2 v(x,y) = 2u(x,y) + 4x\partial_x u(x,y) + (x^2+y^2)\partial_x^2 u(x,y)$$
$$\partial_y^2 v(x,y) = 2u(x,y) + 4y\partial_y u(x,y) + (x^2+y^2)\partial_y^2 u(x,y)$$

Then (we use that $\Delta u = 0$):

$$\Delta v = 4u + 4x\partial_x u + 4y\partial_y u,$$
$$\Delta^2 v = \Delta(4x\partial_x u + 4y\partial_y u)$$

(where we also used that $\Delta u = 0$, and that Δ is a linear operator). Then:

$$\Delta(4x\partial_x u) = \partial_x^2(4x\partial_x u) + \partial_y^2(4x\partial_x u) = \partial_x(4\partial_x u + 4x\partial_x^2 u) + \partial_y^2(4x\partial_x u) =$$

$$= 4\partial_x^2 u + 4\partial_x^2 u + 4x\partial_x^3 u + 4x\partial_x\partial_y^2 u = 8\partial_x^2 u + 4x\partial_x(\Delta u) = 8\partial_x^2 u$$

Similarly,

$$\Delta(4y\partial_y u) = 8\partial_y^2 u,$$

and then

$$\Delta^2 v = 8\Delta u = 0.$$

7. * Solve the following equation for functions $u \in C^2(\mathbb{R}^2, \mathbb{R})$! (10 points)

$$\partial_x u \cdot \partial_y u = 0$$

Solution: This is a bonus problem, the solution can be submitted.