Tenth practice

Eigenvalues, parabolic problems

1. Let $a > 0$, and compute the eigenvalues and eigenfunctions of the following operators!

   a) $D(L) = \{u \in C^2(0,a) \cap C([0,a]) : u(0) = u(a) = 0\}, \quad Lu = -u'',$

   b) $D(L) = \{u \in C^2(0,a) \cap C^1([0,a]) : u'(0) = u'(a) = 0\}, \quad Lu = -u''.$

   Solution:

   a) We seek those numbers $\lambda \in \mathbb{R}$, for which there is such a $u \in D(L), u \not\equiv 0$, for which $Lu = \lambda u$, i.e. $-u'' = \lambda u$. Then by combining these with the conditions inside the domain of the operator, we get the following one-dimensional boundary-value problem:

$$\begin{cases} -u''(x) &= \lambda u(x) \quad (x \in (0,a)) \\ u(0) &= 0 \\ u(a) &= 0. \end{cases}$$

   The solution of this equation is (See Exercise 4 on Practice 1):

$$u(x) = \begin{cases} c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x), & \text{if } \lambda > 0, \\ c_1 e^{\sqrt{|\lambda|} x} + c_2 e^{-\sqrt{|\lambda|} x}, & \text{if } \lambda < 0, \\ c_1 x + c_2, & \text{if } \lambda = 0. \end{cases} \tag{1}$$

   Now let us use the boundary conditions! If $\lambda = 0$, then by $u(0) = 0$ we get $c_2 = 0$, so since $u(a) = 0$ we get $c_1 a = 0$, so $c_1 = 0$, and then $u \equiv 0$. If $\lambda < 0$, then by $u(0) = 0$ we get $c_1 + c_2 = 0$, so since $u(a) = 0$ we get $c_1 e^{\sqrt{|\lambda|} a} - c_1 e^{-\sqrt{|\lambda|} a} = 0$, then $c_1 = 0$, therefore $u \equiv 0$. The remaining case is $\lambda > 0$. In this case since $u(0) = 0$, then $c_2 \cos 0 = 0$, so $c_2 = 0$. On the other hand, since $u(a) = 0$, then $\sin \sqrt{\lambda} a = 0$, therefore $\sqrt{\lambda} a = k\pi$, in which k is a positive whole number (since here we have $\lambda > 0$). This means that $\lambda = \left(\frac{k\pi}{a}\right)^2$, then $u(x) = \sin \frac{k\pi}{a} x$, in which k is a positive whole number, so the eigenvalues are positive, and there are countably infinitely-many of them. It is well know (from e.g. Fourier analysis), that the sinus system is orthogonal in $L^2(0,a)$. Let us norm the previous u functions, then we get the complete system of orthonormal eigenfunctions and eigenvalues of operator L in $L^2(0,a)$:

$$\lambda_k = \left(\frac{k\pi}{a}\right)^2, \qquad u_k(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{k\pi}{a} x\right) \quad (k = 1,2,\dots)$$

   Note that during the normalization, we used that

$$\int_0^a \sin^2\left(\frac{k\pi}{a} x\right) dx = \frac{a}{2},$$

   which can be proved e.g. the following way. If we know that $\cos 2\varphi = \cos^2 \varphi - \sin^2 \varphi$, then we get $\sin^2 \varphi = \frac{1 - \cos 2\varphi}{2}$. Therefore,

$$\int_0^a \sin^2\left(\frac{k\pi}{a} x\right) dx = \int_0^a \frac{1 - \cos\left(\frac{2k\pi}{a} x\right)}{2} dx = \frac{a}{2} - \frac{a}{4k\pi} \left[\sin\left(\frac{2k\pi}{a} x\right)\right]_{x=0}^{a} = \frac{a}{2}.$$

   b) Like in case (a), this problem can also be transformed to a boundary-value problem:

$$\begin{cases} -u''(x) &= \lambda u(x) \quad (x \in (0,a)) \\ u'(0) &= 0 \\ u'(a) &= 0. \end{cases}$$

The solutions of this equation can be found in Table (1). Taking the boundary conditions into account, if $\lambda = 0$ we get $c_1 = 0$, so $u \equiv c_2$. If $\lambda < 0$, then by $u'(0) = 0$ we get $\sqrt{|\lambda|}(c_1 - c_2) = 0$, i.e. $c_1 = c_2$, so by $u'(a) = 0$ we get $\sqrt{|\lambda|}c_1 \left(e^{\sqrt{|\lambda|}a} - e^{-\sqrt{|\lambda|}a}\right) = 0$, therefore $c_1 = 0$, so $u \equiv 0$. Finally, in the case $\lambda > 0$, by $u'(0) = 0$ we get $\sqrt{\lambda}c_1 \cos 0 = = 0$, and then $c_1 = 0$. On the other hand, by $u'(a) = 0$ we get $c_2\sqrt{\lambda}\sin\sqrt{\lambda}a = 0$, consequently $\sqrt{\lambda}a = k\pi$, in which k is a positive whole number (since we consider the case $\lambda > 0$). Then $u(x) = \cos\frac{k\pi}{a}x$, in which by the choice $k = 0$ we get the constant functions we got in the case $\lambda = 0$. The eigenvalues are non-negative, and there are countably infinitely-many of them (also, the 0 eigenvalue has multiplicity one, and the constant functions are the corresponding eigenfunctions). It is well known (from Fourier analysis), that the cosine system is orthogonal in $L^2(0,a)$, so by normalizing the previous functions we get the complete orthonormal eigenfunction and eigenvalue system of operator L inside $L^2(0,a)$:

$$\lambda_k = \left(\frac{k\pi}{a}\right)^2, \qquad u_0(x) = \frac{1}{\sqrt{a}}, \quad u_k(x) = \sqrt{\frac{2}{a}}\cos\left(\frac{k\pi}{a}x\right) \quad (k = 0{,}1{,}2{,}\dots)$$

The constant $\sqrt{\frac{2}{a}}$ comes from similar arguments to the ones discussed in the previous exercise.

2. Let $T = (0,a) \times (0,b) \subset \mathbb{R}^2$ $(a, b > 0)$, and compute the eigenvalues and eigenfunctions of the following operators!

   a) $D(L) = \{u \in C^2(T) \cap C(\overline{T}) : u|_{\partial T} = 0\}, \quad Lu = -\Delta u,$

   b) $D(L) = \{u \in C^2(T) \cap C^1(\overline{T}) : \partial_\nu u|_{\partial T} = 0\}, \quad Lu = -\Delta u.$

   Solution: a) We use the method of separation of variables, i.e. we search for the eigenfunctions in the form $u(x,y) = v(x) \cdot w(y)$. Then the eigenvalue-problem $Lu = \lambda u$ means the following differential equation on the two-dimensional interval T:

$$-v''(x)w(y) - v(x)w''(y) = \lambda v(x)w(y).$$

Supposing that $v(x) \cdot w(y) \neq 0$, after a formal division we get

$$-\frac{v''(x)}{v(x)} = \lambda + \frac{w''(y)}{w(y)}.$$

Note that the left-hand side of the above equation only depends on x, while the right-hand side only depends on y. Since the equation should hold for all $(x,y) \in T$ values, then this can only happen, if we have constant functions on both sides, i.e. there is a constant $\alpha, \beta \in \mathbb{R}$ for which

$$-\frac{v''(x)}{v(x)} = \alpha,$$
$$-\frac{w''(y)}{w(y)} = \beta$$

and $\alpha + \beta = \lambda$. By the boundary conditions we get the homogeneous Dirichlet conditions $v(0) = v(a) = 0$, $w(0) = w(b) = 0$. This means that v is an eigenfunction, and α is an eigenvalue of the operator in Exercise 1. a), so

$$\alpha = \alpha_k = \left(\frac{k\pi}{a}\right)^2$$

and

$$v(x) = v_k(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{k\pi}{a}x\right).$$

Similarly, w is an eigenfunction of the same operator (but here we should write b instead of a), so

$$\beta = \beta_k = \left(\frac{k\pi}{b}\right)^2$$

and

$$w(x) = w_k(y) = \sqrt{\frac{2}{b}} \sin\left(\frac{k\pi}{b} y\right).$$

Consequently, operator L has countably-many eigenvalues, and these are

$$\lambda_{k,l} = \pi^2 \left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right) \quad (k, l = 1,2, \dots),$$

and the corresponding orthonormal eigenfunction-system (which is complete in $L^2(T)$) is

$$u_{k,l}(x, y) = \frac{2}{\sqrt{ab}} \sin\left(\frac{k\pi}{a} x\right) \sin\left(\frac{l\pi}{b} y\right) \quad (k, l = 1,2, \dots).$$

Note that from the previous arguments it is still not clear that the operator has no other eigenvalues. This comes from the fact that the eigenfunctions form a complete orthogonal system (see the Lecture), and the previous system was complete, so there are no other eigenfunctions.

b) We proceed similarly as in the previous exercise. By using the method of separation of variables, we search for our solution in the form $u(x, y) = v(x)w(y)$. After substitution and division (assuming that $v(x)w(y) \neq 0$), we get that

$$-\frac{v''(x)}{v(x)} = \lambda + \frac{w''(y)}{w(y)}.$$

This should hold for all $(x, y) \in T$ values, and it can only hold if we have a constant on both sides, so there exists such an $\alpha, \beta \in \mathbb{R}$ value, for which

$$-\frac{v''(x)}{v(x)} = \alpha,$$

$$-\frac{w''(y)}{w(y)} = \beta$$

and $\alpha + \beta = \lambda$.

Note that $\partial_\nu u|_{\partial T} = -v(x)w'(0)$ on side $(0, a) \times \{0\}$, $\partial_\nu u|_{\partial T} = v(x)w'(b)$ on side $(0, a) \times \{b\}$, $\partial_\nu u|_{\partial T} = -v'(0)w(y)$ on side $\{0\} \times (0, b)$ and $\partial_\nu u|_{\partial T} = v'(a)w(y)$ on side $\{a\} \times (0, b)$. These (under the assumption $v(x)w(y) \neq 0$) mean that $v'(0) = v'(a) = 0$ and $w'(0) = w'(b) = 0$. So we got that v is an eigenfunction, and α is an eigenvalue of the operator L in Exercise 1. b). Similarly, w is an eigenfunction, and β is an eigevalueof the same operator L. By Exercise 1. b), so

$$\alpha = \alpha_k = \left(\frac{k\pi}{a}\right)^2,$$

$$\beta = \beta_k = \left(\frac{k\pi}{b}\right)^2,$$

moreover,

$$v(x) = v_k(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{k\pi}{a} x\right),$$

$$w(x) = w_k(y) = \sqrt{\frac{2}{b}} \cos\left(\frac{k\pi}{b} y\right),$$

in which k, l are non-negative whole numbers. In conclusion, the eigenvalue-system and eigenfunction-system (which is complete in $L^2(T)$) of operator L is:

$$\lambda_{k,l} = \pi^2 \left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right) \quad (k, l = 0,1, \dots),$$

and

$$\begin{aligned}
u_{0,0}(x,y) &= \frac{1}{\sqrt{ab}}, \\
u_{k,0}(x,y) &= \sqrt{\frac{2}{ab}} \cos\left(\frac{k\pi}{a}x\right) \quad (k = 1,2\dots), \\
u_{0,l}(x,y) &= \sqrt{\frac{2}{ab}} \cos\left(\frac{l\pi}{b}y\right) \quad (l = 1,2,\dots), \\
u_{k,l}(x,y) &= \frac{2}{\sqrt{ab}} \cos\left(\frac{k\pi}{a}x\right) \cos\left(\frac{l\pi}{b}y\right) \quad (k,l = 1,2\dots).
\end{aligned}$$

Note that from the previous arguments it is still not clear that the operator has no other eigenvalues. This comes from the fact that the eigenfunctions form a complete orthogonal system (see the Lecture), and the previous system was complete, so there are no other eigenfunctions.

3. Let $T = (0,\pi)^2$ and solve the following elliptic boundary-value problems!

a) $\begin{cases} -\Delta u &= x+y \quad \text{inside } T, \\ u|_{\partial T} &= 0, \end{cases}$

b) $\begin{cases} -\Delta u &= 3\sin x \sin 4y - 8 \sin 2x \sin 5y \quad \text{inside } T, \\ u|_{\partial T} &= 0, \end{cases}$

c) $\begin{cases} -\Delta u &= \cos x \cos y \quad \text{inside } T, \\ \partial_\nu u|_{\partial T} &= 0. \end{cases}$

Solution: a) We construct the solution from the eigenfunction-system of the operator orthonormed in $L^2(T)$. For this, let us write up function f also in this system:

$$f = \sum_{k,l=1}^{\infty} c_{k,l} u_{k,l},$$

in which $u_{k,l}$ is the system from Exercise 2 a). The coefficients can be computed in the following way:

$$\begin{aligned}
c_{k,l} = \int_T f u_{k,l} &= \int_0^\pi \int_0^\pi (x+y) \frac{2}{\pi} \sin(kx)\sin(ly)\,dx\,dy = \\
&= \frac{2}{\pi}\left[\int_0^\pi x\sin(kx)\,dx \int_0^\pi \sin(ly)\,dy + \int_0^\pi y \sin(ly)\,dy \int_0^\pi \sin(kx)\,dx\right] = \\
&= \frac{2}{kl}\left((-1)^{k+1}(1-(-1)^l) + (-1)^{l+1}(1-(-1)^k)\right) =: \frac{2}{kl} w_{k,l},
\end{aligned}$$

in which we used that

$$\int_0^\pi \sin(ly)\,dy = \frac{1}{l}\left[-\cos(ly)\right]_{y=0}^\pi = \frac{1}{l}(1-(-1)^l),$$

and

$$\int_0^\pi x\sin(kx)\,dx = \frac{1}{k}\left[-x\cos(kx)\right]_{x=0}^\pi + \frac{1}{k}\int_0^\pi \cos(ky)\,dy = (-1)^{k+1}\frac{\pi}{k}. \tag{2}$$

By these,

$$u(x,y) = \sum_{k,l=1}^\infty \frac{2w_{k,l}}{kl(k^2+l^2)} \cdot \frac{2}{\pi} \sin(kx)\sin(ly).$$

Note that the convergence of this line is meant in $L^2(T)$ (actually, a much stronger convergence also holds, but those theorems are non-trivial).

b) Since on the right-hand side we have an eigenfunction of the operator on the left, then we should search for the solution in the form

$$u(x,y) = c_1 \sin x \sin 4y + c_2 \sin 2x \sin 5y.$$

Then

$$-\Delta u(x,y) = (3^2+4^2)c_1 \sin 3x \sin 4y + (2^2+5^2)c_2 \sin 2x \sin 5y =$$
$$= 25c_1 \sin 3x \sin 4y + 29c_2 \sin 2x \sin 5y.$$

If we compare these terms to the right-hand side of our equation, we get $25c_1 = 3$, $29c_2 = -8$, so $c_1 = \frac{3}{25}$ and $c_2 = -\frac{8}{29}$. Then the solution of the problem is

$$u(x,y) = \frac{3}{25} \sin x \sin 4y - \frac{8}{29} \sin 2x \sin 5y.$$

By the unique solution of the Dirichlet problem, this is the only solution in $C^2(T) \cap C^1(\overline{T})$.

c) Since on the right-hand side we have an eigenfunction of the operator on the left, then we should search for the solution in the form

$$u(x,y) = c_1 \cos x \cos y + c_2.$$

Then

$$-\Delta u(x,y) = (1^2+1^2)c_1 \cos x \cos y = 2c_1 \cos x \cos y.$$

If we compare these terms to the right-hand side of our equation, we get $2c_1 = 1$, so $c_1 = \frac{1}{2}$. Then the solutions of the problem are

$$u(x,y) = \frac{1}{2} \cos x \cos y + c.$$

By the theorem about the form of the solutions of the Neumann-problem, we get that these are the only solutions in $C^2(T) \cap C^1(\overline{T})$.

4. * Let $T := (0,\pi)^2 \subset \mathbb{R}^2$, and also $\Gamma_1 := \{\pi\} \times [0,\pi)$, $\Gamma_2 := (0,\pi] \times \{\pi\}$, $\Gamma_3 := \{0\} \times (0,\pi]$, $\Gamma_4 := [0,\pi) \times \{0\}$. Also, let $g, h : \mathbb{R}^2 \to \mathbb{R}$ be such functions that

$$h|_{\Gamma_1 \cup \Gamma_3} = 1, \quad h|_{\Gamma_2 \cup \Gamma_4} = 0,$$
$$g|_{\Gamma_1 \cup \Gamma_3} = 0, \quad g|_{\Gamma_2 \cup \Gamma_4} = 1.$$

Then solve the following boundary-value problem! (12 points)

$$\begin{cases} -\Delta u(t,x) = \sin(x)\cos(3y) - 5\sin(3x)\cos(4y) & \text{inside } T, \\ (g\partial_\nu u + hu)|_{\partial T} = 0. \end{cases}$$

Solution: The solution can be submitted.

5. Solve the following mixed parabolic problems!

a) $$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) &= 0 & ((t,x) \in (\mathbb{R}^+ \times (0,\pi))), \\ u(0,x) &= x & (x \in [0,\pi]), \\ u(t,0) = u(t,\pi) &= 0 & (t \in \mathbb{R}_0^+). \end{cases}$$

b) $$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) &= 0 & ((t,x) \in (\mathbb{R}^+ \times (0,\pi))), \\ u(0,x) &= \sin 3x - 4\sin 5x & (x \in [0,\pi]), \\ u(t,0) = u(t,\pi) &= 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

c) $$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) &= \sin 2x & ((t,x) \in (\mathbb{R}^+ \times (0,\pi))), \\ u(0,x) &= 0 & (x \in [0,\pi]), \\ u(t,0) = u(t,\pi) &= 0 & (t \in \mathbb{R}_0^+). \end{cases}$$

d) $$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) &= t\sin x & ((t,x) \in (\mathbb{R}^+ \times (0,\pi))), \\ u(0,x) &= 0 & (x \in [0,\pi]), \\ u(t,0) = u(t,\pi) &= 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

Solution: a) We use Fourier's method: we search for the solution u in the form

$$u(t,x) = \sum_{k=1}^{\infty} \xi_k(t) u_k(x),$$

in which u_k is the eigenfunction of the one-dimensional (minus) Laplace operator with Dirichlet boundary ($k = 1, \dots$) and $\xi_k(t)$ are some unknown functions depending only on t. For this, let us write up the functions present in the equation also in the $\{u_k\}_{k=1}^{\infty}$ basis. The series of the constant 0 function is easy, since all of the coefficients in its series are zero. It is also clear that

$$x = \sum_{k=1}^{\infty} \left(\int_0^{\pi} x \sin kx \, dx \right) \cdot u_k(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi}{k} u_k(x),$$

in which we used the fact (2) from Exercise 3. a), so

$$\int_0^{\pi} x \sin kx \, dx = (-1)^{k+1} \frac{\pi}{k}.$$

Also, since $u(t, x) = \sum_{k=1}^{\infty} \xi_k(t) u_k(x)$, we have

$$\partial_t u(t, x) = \sum_{k=1}^{\infty} \xi_k'(t) u_k(x)$$

and

$$\partial_x^2 u(t, x) = \sum_{k=1}^{\infty} \xi_k(t) u_k''(x) = \sum_{k=1}^{\infty} \xi_k(t) \left(\sqrt{\frac{2}{\pi}} \sin(kx) \right)'' = -k^2 \sum_{k=1}^{\infty} \xi_k(t) u_k(x)$$

If we substitute these into our equation, we get

$$\sum_{k=1}^{\infty} \xi_k'(t) u_k(x) + k^2 \sum_{k=1}^{\infty} \xi_k(t) u_k(x) = 0,$$

which can only hold if $\xi_k'(t) + k^2 \xi_k(t) = 0$. Similarly, for the initial condition we have

$$\sum_{k=1}^{\infty} \xi_k(0) u_k(x) = u(0, x) = x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi}{k} u_k(x),$$

which can only hold if $\xi_k(0) = (-1)^{k+1} \frac{\pi}{k}$.

Then, we get the following initial-value problem for ξ_k inside $\mathbb{R}^+$:

$$\begin{cases} \xi_k'(t) + k^2 \xi_k(t) = 0, \\ \qquad \xi_k(0) = (-1)^{k+1} \dfrac{\pi}{k}. \end{cases}$$

Let us multiply both sides of the equation by $e^{k^2 t}$, then

$$\xi_k'(t) e^{k^2 t} + k^2 e^{k^2 t} = 0,$$

$$\left(\xi_k(t) e^{k^2 t} \right)' = 0,$$

$$\xi_k(t) = \xi_k(0) e^{-k^2 t} = (-1)^{k+1} \frac{\pi}{k} e^{-k^2 t}.$$

Therefore, the solution of the parabolic mixed problem is:

$$u(t, x) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi}{k} e^{-k^2 t} \sin kx.$$

Note that the above convergence meant for every $t > 0$ inside $L^2(0, \pi)$ (actually, a much stronger convergence also holds, but those theorems are non-trivial).

b) We use Fourier's method: we search for the solution u in the form

$$u(t, x) = \sum_{k=1}^{\infty} \xi_k(t) \sin kx,$$

since the initial function is the eigenfunction of the one-dimensional (minus) Laplace operator with homogeneous Dirichlet boundary. Then by the equation and the conditions we get that

$$\xi_k' + \xi_k = 0,$$
$$\xi_k(0) = 0,$$

if $k \neq 3, 5$, which has only the constant zero function as a solution, and also

$$\xi_3'(t) = -9\xi_3(t), \ \xi_3(0) = 1$$

and

$$\xi_5'(t) = -25\xi_5(t), \ \xi_5(0) = -4.$$

The solutions of these are $\xi_3(t) = e^{-9t}$ and $\xi_5(t) = -4e^{-25t}$, so the solution of the parabolic mixed problem is

$$u(t,x) = e^{-9t}\sin 3x - 4e^{-25t}\sin 5x.$$

There are no other solutions, since the solution of the mixed problem is unique.

c) We use the method of Fourier: let us write function $\sin 2x$ into the form

$$\sin 2x = \sum_{k=1}^{\infty} c_k \cdot \sin kx$$

It is clear that $c_k = 1$ if $k = 2$, and in other cases $c_k = 0$. Then by searching for solution u in the form

$$u(t,x) = \sum_{k=1}^{\infty} \xi_k(t)\sin kx,$$

by the equation and the conditions we get that $\xi_k = 0$, if $k \neq 2$, and for ξ_2:

$$\xi_2'(t) + 4\xi_2(t) = 1, \ \xi_2(0) = 0.$$

By multiplying both sides of the equation with e^{4t}:

$$e^{4t}\xi_2'(t) + 4e^{4t}\xi_2(t) = e^{4t},$$
$$(e^{4t}\xi_2(t))' = e^{4t},$$
$$\xi_2(t) = e^{-4t}C + \frac{1}{4},$$

and since $\xi_2(0) = C + \frac{1}{4} = 0$, $C = -\frac{1}{4}$ and we have $\xi_2(t) = \frac{1}{4}\left(1 - e^{-4t}\right)$. Consequently, the solution of the mixed parabolic problem is

$$u(t,x) = \frac{1}{4}(1 - e^{-4t})\sin 2x.$$

There are no other solutions, since the solution of the mixed problem is unique.

d) As in the previous cases, we apply the Fourier's method to the problem. We seek our solution u in the form

$$u(t,x) = c(t)\sin x.$$

Then from the initial condition we get $c(0) = 0$, and by substituting u into the equation we get the ordinary differential equation

$$c'(t) + c(t) = t.$$

By multiplying both sides by e^t,

$$e^t c'(t) + e^t c(t) = te^t$$
$$(e^t c(t))' = te^t,$$

so (since $((te^t - e^t)' = te^t)$:

$$c(t) = t - 1 + ce^{-t}$$

and by the initial condition

$$c(t) = t - 1 + e^{-t}.$$

The solution of the parabolic mixed problem is

$$u(t,x) = (e^t + t - 1)\sin x.$$

There are no other solutions, since the solution of the mixed problem is unique.