# Second week

## First order equations

I. **First order, homogeneous linear equations.**

These equations have the form

$$\sum_{i=1}^{n} f_i(x)\frac{\partial u(x)}{\partial x_i} = 0 \tag{1}$$

The **characteristic equation** of (1) is:

$$\dot{x}(t) = f(x(t)) \tag{2}$$

where $f = (f_1, \dots f_n) : \mathbf{R}^n \to \mathbf{R}^n$). The solutions of (2) are the **characteristics** of (1).

**Theorem:** Then the following two statements are equivalent:

- $u$ is a solution of (1).
- $u$ is constant along the characteristics (the solutions of (2)).

1. We seek the classical solutions of the following equations:

(a) $y\ \partial_x u(x,y) - x\ \partial_y u(x,y) = 0$

**Solution:** Then the characteristic equation is

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t). \end{cases} \tag{3}$$

From now on, we can proceed further in two different ways.

*First method:* The solution of (3) is $x(t) = c_1 \sin(t) + c_2 \cos(t)$ and $y(t) = c_1 \cos(t) - c_2 \sin(t)$. Then it is easy to see that

$$(x(t))^2 + (y(t))^2 = \text{ const.}$$

which means that $\phi(x,y) = x^2 + y^2$ is a first integral of (3), meaning that it is constant along the characteristics. However we know that $u$ also has this property, and thus

$$u(x,y) = \Phi(x^2 + y^2)$$

for some $\Phi \in C^1$ function.

*Second method:* We do not need to solve (3): we only have to realize that

$$\dot{x}(t)\ x(t) + \dot{y}(t)\ y(t) = 0,$$

which means

$$\left(\frac{1}{2}(x(t))^2 + \frac{1}{2}(y(t))^2\right)' = 0.$$

So $(x(t))^2 + (y(t))^2$ is constant along the solutions of (3), which means that

$$u(x,y) = \Phi(x^2 + y^2)$$

for some $\Phi \in C^1$ function.

(b) $x\ \partial_x u(x,y) = y\ \partial_y u(x,y)$

**Solution:** Then the characteristic equation is

$$\begin{cases} x'(t) = x(t), \\ y'(t) = -y(t). \end{cases} \tag{4}$$

From now on, we can proceed further in two different ways.

*First method:* The solution of (4) is $x(t) = c_1 e^t$ and $y(t) = c_2 e^{-t}$. Then it is easy to see that

$$x(t)y(t) = \text{ const.}$$

which means that $\phi(x,y) = xy$ is a first integral of (4), meaning that it is constant along the characteristics. However we know that $u$ also has this property, and thus

$$u(x,y) = \Phi(xy)$$

for some $\Phi \in C^1$ function.

*Second method:* We do not need to solve (4): we only have to realize that

$$\dot{x}(t)\ y(t) + \dot{y}(t)\ x(t) = 0,$$

which means

$$(x(t)\ y(t))' = 0.$$

So $x(t)y(t)$ is constant along the solutions of (4), which means that

$$u(x,y) = \Phi(xy)$$

for some $\Phi \in C^1$ function.

**Remark:** The curve $xy = c$ has two branches, and the solution is not necessarily the same on these branches: it might happen that $u(-1,-1) \neq u(1,1)$, while they correspond to the same, $xy = 1$ equation (but are on different branches).

(c) $\partial_x u(x,y) = y \partial_y u(x,y)$

**Solution:** Then the characteristic equation is

$$\begin{cases} x'(t) = 1, \\ y'(t) = -y(t). \end{cases} \tag{5}$$

The solution of (5) is $x(t) = t + c_1$ and $y(t) = c_2 e^{-t}$. Then it is easy to see that

$$e^{x(t)} y(t) = c_2 e^{c_1} = \text{ const,}$$

which means that $\phi(x,y) = e^x y$ is a first integral of (5), meaning that it is constant along the characteristics. However we know that $u$ also has this property, and thus

$$u(x,y) = \Phi(e^x y)$$

for some $\Phi \in C^1$ function.

(d) $y^2 \partial_x u(x,y) + e^x \partial_y u(x,y) = 0$

**Solution:** Then the characteristic equation is

$$\begin{cases} x'(t) = (y(t))^2, \\ y'(t) = e^{x(t)}. \end{cases} \tag{6}$$

We do not need to solve (6): we only have to realize that

$$\dot{y}(t)\ (y(t))^2 = e^{x(t)} \dot{x}(t),$$

which means

$$\dot{y}(t)\ (y(t))^2 - e^{x(t)} \dot{x}(t) = 0,$$
$$\left( \frac{(y(t))^3}{3} - e^{x(t)} \right)' = 0.$$

So $\dfrac{(y)^3}{3} - e^x$ is constant along the solutions of (6), which means that

$$u(x,y) = \Phi\left( \frac{(y)^3}{3} - e^x \right)$$

for some $\Phi \in C^1$ function.

(e) $yz\,\partial_x u(x,y,z) + xz\,\partial_y u(x,y,z) + (x^2+y^2)\partial_z u(x,y,z) = 0$

**Solution:** Then the characteristic equation is

$$\begin{cases} x'(t) = y(t)z(t), \\ y'(t) = x(t)z(t), \\ z'(t) = (x(t))^2 + (y(t))^2. \end{cases} \tag{7}$$

We do not need to solve (7): we only have to realize that

$$\dot{x}(t)\,x(t) - \dot{y}(t)\,y(t) = 0,$$

which means that $\varphi_1(x,y,z) = x^2 - y^2$ is a first integral. Also,

$$\dot{z}(t)\,z(t) = (x(t))^2 z(t) + (y(t))^2 z(t) = x(t)\dot{y}(t) + y(t)\dot{x}(t) = (x(t)y(t))',$$

which means that

$$\left(\frac{(z(t))^2}{2} - x(t)y(t)\right)' = 0.$$

So $\varphi(x,y,z) = \frac{z^2}{2} - xy$ is also a first integral of (7), and it is independent of $\varphi_1$, which means that our solution is

$$u(x,y) = \Phi\left(x^2 - y^2, \frac{z^2}{2} - xy\right)$$

for some $\Phi \in C^1$ function.

(f) $x\partial_x u(x,y,z) + y\partial_y u(x,y,z) + z\partial_z u(x,y,z) = 0$

**Solution:** The characteristic equation is

$$\begin{cases} x'(t) = x(t), \\ y'(t) = y(t), \\ z'(t) = z(t). \end{cases} \tag{8}$$

The solution of is $x(t) = c_1 e^t$, $y(t) = c_2 e^t$ and $z(t) = c_3 e^t$. Then we have to realize that

$$\frac{x(t)}{y(t)} = \frac{c_1}{c_2} \qquad \text{and} \qquad \frac{y(t)}{z(t)} = \frac{c_2}{c_3},$$

which means that the two first integrals are

$$\phi_1(x,y,z) = \frac{x}{y} \qquad \text{and} \qquad \phi_2(x,y,z) = \frac{y}{z}.$$

Since these two are independent, then our solution is

$$u(x,y) = \Phi\left(\frac{x}{y}, \frac{y}{z}\right)$$

for some $\Phi \in C^1$ function.

Note that the Cauchy problems corresponding to the homogeneous equations are in exercise 3. after the next section.

II. **Quasi-linear equations** These equations have the form

$$\sum_{i=1}^{n} f_i(x, u(x)) \frac{\partial u(x)}{\partial x_i} = f_0(x, u(x)) \tag{9}$$

Then the **auxiliary equation** of (9) is:

$$\sum_{i=1}^{n} f_i(x, u(x)) \frac{\partial v(x,u)}{\partial x_i} + f_0(x, u(x)) \frac{\partial v(x,u)}{\partial u} = 0 \tag{10}$$

The main idea of these exercises is observe the characteristic equation associated with (10) (since it is also a homogeneous linear equation). Then search for two different first integrals, and use the fact that for two different (independent) first integrals $\varphi_1$ and $\varphi_2$, the connection $\varphi_1 = \Psi(\varphi_2)$ holds. From this we can have the value of $u$.

2. Give the $u \in C^2(\mathbf{R^3})$ solution to the following equations!

a) $y\ u(x,y)\partial_x u(x,y) + x\ u(x,y)\partial_y u(x,y) = x^2 + y^2$

**Solution:** The auxiliary equation is

$$y\ u(x,y)\partial_x v(x,y,u) + x\ u(x,y)\partial_y v(x,y,u) + (x^2+y^2)\partial_u v(x,y,u) = 0. \tag{11}$$

Then the characteristic equation associated with (11) is

$$\begin{cases} x'(t) = y(t)\hat{u}(t), \\ y'(t) = x(t)\hat{u}(t), \\ \hat{u}'(t) = (x(t))^2 + (y(t))^2, \end{cases} \tag{12}$$

where $\hat{u}(t) := u(x(t), y(t))$. Note that this is the same as the characteristic equation in exercise 1. e), so the two first integrals are $\varphi_1(x,y,z) = x^2 - y^2$ and $\varphi(x,y,z) = \dfrac{\hat{u}^2}{2} - xy$. Then because of the connection between them:

$$\frac{u^2}{2} - xy = \Psi(x^2 - y^2).$$

From which we get that

$$u(x,y) = \pm\sqrt{2xy + \Psi(x^2 - y^2)}.$$

b) $y\ \partial_x u(x,y) - x\ \partial_y u(x,y) = 2xyu(x,y)$

**Solution:** The auxiliary equation is

$$y\ \partial_x v(x,y,u) - x\ \partial_y v(x,y,u) + 2xyu\partial_u v(x,y,u) = 0. \tag{13}$$

Then the characteristic equation associated to (11) is

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t), \\ \hat{u}'(t) = 2x(t)y(t)\hat{u}(t), \end{cases} \tag{14}$$

where $\hat{u}(t) := u(x(t), y(t))$. We don't have to solve it, just find two independent first integrals of the equation. From the first two equation it is clear that $\varphi_1(x,y,u) = x^2 + y^2$ is a first integral. Also,

$$\frac{\hat{u}'(t)}{\hat{u}(t)} = 2x(t)y(t) = 2x(t)x'(t).$$

From which

$$\frac{\hat{u}'(t)}{\hat{u}(t)} - 2x(t)x'(t) = 0,$$

$$\left(\ln|\hat{u}(t)| - (x(t))^2\right)' = 0.$$

So the other first integral is $\varphi_2(x,y,u) = \ln|\hat{u}(t)| - (x(t))^2$. Then because of the connection between the two first integrals:

$$\ln|\hat{u}(t)| - (x(t))^2 = \Psi(x^2 + y^2),$$

from which we get that

$$u(x,y) = \pm e^{x^2 + \Psi(x^2+y^2)}.$$

Note that since $\Phi$ is arbitrary, we might write

$$u(x,y) = \pm e^{x^2}\tilde{\Psi}(x^2 + y^2).$$

3. Solve the following Cauchy problems!

(a)

$$\begin{cases} x\partial_x u(x,y) - y\partial_y u(x,y) = 0, \\ u\left(x, \dfrac{1}{x}\right) = 1 \end{cases} \tag{15}$$

**Solution:** Because of exercise 1. b), the solution is constant on the $xy = c$ hyperbolas. By $u\left(x, \dfrac{1}{x}\right) = 1$, we give the value on one of the characteristics - however, the values on all the others are unknown, meaning that we have infinitely many solutions.

(b)

$$\begin{cases} x\partial_x u(x,y) - y\partial_y u(x,y) = 0, \\ u\left(x, \dfrac{1}{x}\right) = x. \end{cases} \tag{16}$$

**Solution:** Because of exercise 1. b), the solution is constant on the $xy = c$ hyperbolas. By $u\left(x, \frac{1}{x}\right) = x$, we require that $u = x$ on the curve $y = \frac{1}{x}$, but this is a characteristic, so it should be constant, meaning that the equation has no solution.

(c)

$$\begin{cases} x\partial_x u(x,y) - y\partial_y u(x,y) = 0, \\ u(x,y) = u(-x,-y), \\ u\left(x, x^2\right) = x. \end{cases} \tag{17}$$

**Solution:** Because of $u(x,y) = u(-x,-y)$, it is enough to determine the solution on one of the half-planes. We know from exercise 1. b) that $u(x,y) = \phi(xy)$, meaning that $u(x,x^2) = = \phi(x^3) = x$. If we use a new variable $w := x^3$, then $x = \sqrt[3]{w}$ and $\phi(w) = \sqrt[3]{w}$, which means that $u(x,y) = \sqrt[3]{xy}$.

**Remark:** Here the line at which the initial condition is given ($y = x^2$) intersects all of the characteristics exactly once, so this is the reason why we have exactly one solution here.

(d)

$$\begin{cases} xu(x,y)\partial_x u(x,y) + xu(x,y)\partial_y u(x,y) = x^2 + y^2, \\ u(x,0) = x^2. \end{cases} \tag{18}$$

**Solution:** From 2. (d) we know that $u(x,y) = \pm\sqrt{2xy + \Psi(x^2 - y^2)}$. Then

$$u(x,0) = \pm\sqrt{\Psi(x^2)} = x^2,$$

$$\sqrt{\Psi(x^2)} = x^2,$$

$$\Psi(x^2) = x^4,$$

$$\Psi(x) = x^2,$$

and consequently $u(x,y) = \sqrt{2xy + (x^2 - y^2)^2}$.

(e)

$$\begin{cases} x\partial_x u(x,y) + y\partial_y u(x,y) = u(x,y), \\ u(x,1) = x^2. \end{cases} \tag{19}$$

**Solution:** The auxiliary equation is

$$x\ \partial_x v(x,y,u) + y\ \partial_y v(x,y,u) + u\partial_u v(x,y,u) = 0. \tag{20}$$

Then the characteristic equation associated with (20) is

$$\begin{cases} x'(t) = x(t), \\ y'(t) = y(t), \\ \hat{u}'(t) = \hat{u}(t). \end{cases} \tag{21}$$

Then from 1. (f) we know that the two first integrals are $\phi_1(x,y,u) = \frac{u}{y}$ and $\phi_2(x,y,u) = \frac{x}{y}$. Then $\frac{u}{y} = \Psi\left(\frac{x}{y}\right)$, and consequently

$$u(x,y) = y\ \Psi\left(\frac{x}{y}\right).$$

Using the condition:

$$u(x,1) = \Phi(x) = x^2,$$

and

$$u(x,y) = \frac{x^2}{y}.$$

(f)

$$\begin{cases} x\partial_x u(x,y) - \partial_y u(x,y) = 1, \\ u(x,0) = x. \end{cases} \tag{22}$$

**Solution:** The auxiliary equation is

$$x\,\partial_x v(x,y,u) - \partial_y v(x,y,u) + \partial_u v(x,y,u) = 0. \tag{23}$$

Then the characteristic equation associated with (23) is

$$\begin{cases} x'(t) = x(t), \\ y'(t) = -1, \\ \hat{u}'(t) = 1. \end{cases} \tag{24}$$

Then it is clear that $(u(t) + y(t))' = 0$, so $\phi_1(x,y,u) = u + y$ is a first integral. We also know that $x(t) = c_1 e^t$ and $y(t) = -t + c_2$, so

$$x(t)e^{y(t)} = c_1 e^{c_2},$$

which means that $\phi_2(x,y,u) = xe^y$ is also a first integral. Consequently,

$$u + y = \Psi(xe^y),$$

and $u(x,y) = -y + \Psi(xe^y)$. Then using the initial condition:

$$u(x,0) = \Phi(x) = x,$$

and then $u(x,y) = xe^y - y$.

4. Is there such a classical solution of the following equation, for which $u(0,y) = y$?

$$y\,\partial_x u(x,y) - x\partial_y u(x,y) = y.$$

**Solution:** The auxiliary equation is

$$y\,\partial_x v(x,y,u) - x\,\partial_y v(x,y,u) + y\,\partial_u v(x,y,u) = 0. \tag{25}$$

Then the characteristic equation associated with (25) is

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t), \\ \hat{u}'(t) = y(t). \end{cases} \tag{26}$$

Then it is clear from the first two equations that $\phi_1(x,y,u) = x^2 + y^2$ is a first integral. Also, $u'(t) - x'(t) = 0$, so $\phi_2(x,y,u) = u - x$ is also a first integral. Consequently,

$$u - x = \Psi(x^2 + y^2),$$

$$u(x,y) = x + \Psi(x^2 + y^2).$$

Then $u(0,y) = \Psi(y^2) = y$. However, if $y = 1$, then $\Psi(1) = 1$, but if $y = -1$, then $\Psi((-1)^2) = \Psi(1) = -1$, which is a contradiction, meaning that there is no such solution.

**Remark:** Here there is no global solution, but there are local solutions for the cases $y < 0$ and $y > 0$.

5. Which solution of the following equation is tangent to the $y$-axis?

$$y\partial_x u(x,y) - x\partial_y u(x,y) = x^3 y + xy^3$$

**Solution:** The auxiliary equation is

$$y\,\partial_x v(x,y,u) - x\,\partial_y v(x,y,u) + (x^3 y + xy^3)\,\partial_u v(x,y,u) = 0. \tag{27}$$

Then the characteristic equation associated with (27) is

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t), \\ \hat{u}'(t) = x^3 y + xy^3. \end{cases} \tag{28}$$

Then it is clear from the first two equations that $\phi_1(x,y,u) = x^2 + y^2$ is a first integral. Also,

$$u'(t) + y'(t)(y(t))^3 - x'(t)(x(t))^3 = 0,$$

so $\phi_2(x,y,u) = u + \frac{y^4}{4} - \frac{x^4}{4}$. Consequently,

$$u + \frac{y^4}{4} - \frac{x^4}{4} = \Psi(x^2+y^2),$$

$$u(x,y) = -\frac{y^4}{4} + \frac{x^4}{4} + \Psi(x^2+y^2).$$

Our solution is tangent to the $y$-axis, if $u(0,y) = 0$ for some $y$. Then

$$u(0,y) = -\frac{y^4}{4} + \Psi(y^2) = 0,$$

and $\Psi(y) = \frac{y^2}{4}$. So our solution is

$$u(x,y) = \frac{(x^2+y^2)^2}{4} - \frac{y^4}{4} + \frac{x^4}{4} = \frac{x^4}{2} + \frac{x^2y^2}{2}.$$

6. Let $H \in C^1(\mathbf{R}^2)$, $f, g \in C(\mathbf{R})$. Give the first integral of the following systems!

(a)

$$\begin{cases} x'(t) = \partial_y H(x,y), \\ y'(t) = -\partial_x H(x,y). \end{cases} \tag{29}$$

**Solution:** Here $H$ is the *Hamiltonian function* (or just simply *Hamiltonian*), and it is a first integral, since

$$(H(x(t),y(t)))' = \partial_x H(x,y)x'(t) + \partial_y H(x,y)y'(t) = -\partial_x H(x,y)\partial_y + \partial_y\partial_x = 0.$$

(b)

$$\begin{cases} x'(t) = f(y), \\ y'(t) = g(x). \end{cases} \tag{30}$$

**Solution:** If $H(x,y) = F(y) - G(x)$, then $\partial_y H(x,y) = F'(y) = f(y)$ and $\partial_x H(x,y) = -G'(x) = -g(x)$, so we got back the equation of exercise (a), which means that this is just a special case of that one.

**Remark:** The previous argument shows that if our equation is in the form

$$f(y)\partial_x u(x,y) + g(y)\partial_y u(x,y) = 0,$$

then the solution is $u(x,y) = \Phi(F(y) - G(x))$.

7. * Solve the following Cauchy problem! (10 points)

$$\begin{cases} xu(x,y)\partial_x u(x,y) + yu(x,y)\partial_y u(x,y) = x^2 + y^2 + (u(x,y))^2, \\ u(1,y) = y^2. \end{cases} \tag{31}$$

**Solution:** This is a bonus problem, the solution can be submitted.