Third week

Classification of second order equations, canonical form

1. Categorize the following differential operators!

   (a) $Lu = \partial_x^2 u + 6\partial_{xy} u + \partial_y^2 u$
   Solution: The matrix is
   $$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$
   In this case $\det(A) = -8 < 0$, so one of the eigenvalues is negative, and the other is positive (since $\lambda_1\lambda_2 = \det(A)$), so the operator is hyperbolic.

   (b) $Lu = 6\partial_x^2 u + 8\partial_{xy} u + 8\partial_y^2 u + 2\partial_{xz} u + 6\partial_{yz} u + 10\partial_z^2 u$
   Solution: The matrix is
   $$A = \begin{pmatrix} 6 & 4 & 1 \\ 4 & 8 & 3 \\ 1 & 3 & 10 \end{pmatrix}$$
   Now we can proceed further in two different ways.
   Method one: We can use the **theorem of Gershgorin**: this states that all of the eigenvalues of a matrix lie inside the union of some circles, which have their centers located at points a_{ii}, and the corresponding radiuses are $\sum_{j=1, j\neq i}^n |a_{ij}|$. In this case this means that all of the eigenvalues have to be positive (since all of the circles lie on the positive side of the real number line), so the operator is elliptic.
   Method two: Since all of the principal minors are positive for the above matrix, then it is positive definite, meaning that all of its eigenvalues are positive, so it is elliptic.
   Alternatively one can also compute the eigenvalues of the given matrix, but that takes more time.

   (c) $Lu = (x+y)\partial_x^2 u + 2\sqrt{xy}\partial_{xy} u + (x+y)\partial_y^2 u$
   Solution: The matrix is
   $$A = \begin{pmatrix} x+y & \sqrt{xy} \\ \sqrt{xy} & x+y \end{pmatrix}$$
   Here $\det(A) = (x+y)^2 - xy = x^2 + xy + y^2$. However, since we need the equation to be well-defined, $xy \geq 0$ (since we only take account real-valued coefficients), so it means that $\det(A) = x^2 + xy + y^2 \geq 0$. Also, the case $\det(A) = 0$ can only hold if $x = y = 0$, but in this case we get the identically zero operator, so we can omit this case. Consequently, our operator is elliptic for $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

2. Show such a differential operator which is elliptic on $\mathbb{R}^+ \times \mathbb{R}^+$, and hyperbolic on $\mathbb{R}^- \times \mathbb{R}^+$ and on $\mathbb{R}^+ \times \mathbb{R}^-$. Is it true that all such operators are parabolic on $\mathbb{R}^+ \times \{0\}$?

   Solution: A good example is $x\partial_x^2 u + y\partial_y^2 u$, since then the matrix has the form
   $$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$
   Then it has the required properties.
   If the elements in the matrix are not continuous functions, then we can easily define such functions which take positive values for $y \geq 0$ and negatives for $y < 0$, so it will not be parabolic on $\mathbb{R}^+ \times \{0\}$. However, if the coefficient functions are continuous, then the operator will be parabolic on $\mathbb{R}^+ \times \{0\}$: the reason is that the function $(x_0, y) \to \min\{\text{eig}(A(x_0, y))\}$ is continuous in y. (The reason is that the roots of a polynomial depend continuously on its coefficients - this can be proved e.g. by Rouche's theorem.)

3. Show an operator which is elliptic on all points of $\mathbb{R}^n$, but not uniformly elliptic.

   Solution: Let $A(x)$ be a diagonal matrix with all its entries as e^{-x_1}. Then for all $x \in \mathbb{R}^n$, $\forall p \in \mathbb{R}^n \setminus \{0\}$:
   $$\langle A(x)p, p\rangle = e^{-x_1}|p|^2 > 0$$
   Which means that the operator is elliptic.
   However, since $e^{-x_1}|p|^2 \to 0$ as $x_1 \to \infty$, it is not uniformly elliptic (since it is not bounded from below).

4. Is it possible to give a differential operator with continuous coefficients which is elliptic in all inner points of a bounded set $\Omega \subset \mathbb{R}^n$, but is not uniformly elliptic? And what if the functions are continuous on $\overline{\Omega}$, and the operator is also elliptic on $\overline{\Omega}$?

   Solution: A good example for the first question is for example $\Omega = (0,1)^n$ and $A(x) = \text{diag}(x_1, x_2, \dots x_n)$. In this case

$$\langle A(x)p, p\rangle = x_1|p_1|^2 + x_2|p_2|^2 + \dots x_n|p_n|^2 > 0$$

   So our operator is elliptic. However, if $x \to 0$, then $\langle A(x)p, p\rangle \to 0$, so it is not uniformly elliptic (it is not bounded from below).

   Now we prove that if the operator is defined on $\overline{\Omega}$, then the elliptic property implies the uniform elliptic one. Let us consider the surface of $B(0,1)$ (the ball with radius one centered at zero), and denote it by S. Then we know that the function $(x,p) \to \langle A(x)p, p\rangle$ is continuous on $\overline{\Omega} \times S$. Because of the elliptic property, we know that $\langle A(x)p, p\rangle > 0$. Since $\overline{\Omega} \times S$ is compact, we know that there is some $c_0 > 0$ constant for which $\langle A(x)p, p\rangle \geq c_0$ for all $(x,p) \in \overline{\Omega} \times S$.

   If p is arbitrary ($p \in \mathbb{R}^n \setminus \{0\}$), then $\frac{p}{|p|} \in S$, meaning that

$$\langle A(x)p, p\rangle = |p|^2 \left\langle A(x)\frac{p}{|p|}, \frac{p}{|p|}\right\rangle \geq c_0|p|^2$$

   so it is uniformly elliptic. (It can be shown easily that if this inner product is bounded from below, then it is also bounded from above, so that part can be omitted.)

5. Give $a, b, c : \mathbb{R}^2 \to \mathbb{R}$ polynomials such that the differential operator

$$Lu = a(x,y)\partial_x^2 u + b(x,y)\partial_{xy} u + c(x,y)\partial_y^2 u$$

   is elliptic on the upper (open) half plane, and hyperbolic on the lower (open) half plane.

   Solution:
   First Solution: Let e.g. $a(x,y) = y^2$, $b(x,y) = 2xy$ and $c(x,y) = x^2 + y$. Then

$$A = \begin{pmatrix} y^2 & xy \\ xy & x^2 + y \end{pmatrix}$$

   Here $\det(A) = y^2x^2 + y^3 - x^2y^2 = y^3$. This is positive for $y > 0$ and negative for $y < 0$, so the matrix is elliptic for $y > 0$ and hyperbolic for $y < 0$.

   Second Solution: Let our matrix to be

$$A = \begin{pmatrix} 1 + y & y \\ y & y \end{pmatrix}$$

   Here $\det(A) = y + y^2 - y^2 = y$. This is positive for $y > 0$ and negative for $y < 0$, so the matrix is elliptic for $y > 0$ and hyperbolic for $y < 0$.

   Of course there are several other possible configurations.

6. Give the a, b non-constant polynomials in a way that the differential operator

$$Lu = a(x,y)\partial_x^2 u + x^2\partial_{xy} u + y^2\partial_{yx} u + b(x,y)\partial_y^2 u$$

   is elliptic inside $B(0,1)$ and inside $\mathbb{R}^2 \setminus \overline{B(0,2)}$, and hyperbolic inside $B(0,2) \setminus \overline{B(0,1)}$.

   Solution: Here

$$A = \begin{pmatrix} a(x+y) & \frac{1}{2}(x^2+y^2) \\ \frac{1}{2}(x^2+y^2) & b(x,y) \end{pmatrix}$$

   Then

$$\det(A) = a(x,y)b(x,y) - \frac{1}{4}(x^2+y^2)^2 \tag{1}$$

   Our goal is now that the determinant is negative at $B(0,2) \setminus \overline{B(0,1)}$, and is positive in $B(0,1)$ and in $\mathbb{R}^2 \setminus \overline{B(0,2)}$. Note however that the previous conditions can be easily expressed by polar-coordinates: namely, we want our determinant to be positive for $r^2 < 1$ and $r^2 > 4$ and to be negative for $1 <$ $< r^2 < 4$. (Note that it is possible to do the following calculations using only r, but it is easier in this case to use r^2 instead.)

Then $(r^2-1)(r^2-4)$ is a good choice, so let us choose our determinant to be $\det(A) = \frac{1}{2}(r^2-1)(r^2-4)$. (The reason for the constant $\frac{1}{2}$ is that we want to have a product form $a(x,y)b(x,y)$ in the end - it might work with other constants as well, but does not work with 1.)

Substituting this into (1), we get

$$a(x,y)b(x,y) = \det(A) + \frac{1}{4}(x^2+y^2)^2 = \frac{1}{2}(r^2-1)(r^2-4) + \frac{1}{4}r^4 =$$
$$\frac{3}{4}r^4 - \frac{5}{2}r^2 + 2 = \frac{1}{4}(3r^4 - 10r^2 + 8) = \frac{1}{4}(3r^4 - 6r^2 - 4r^2 + 8) =$$
$$= \frac{1}{4}(r^2-2)(3r^2-4)$$

So the choices $a(x,y) = \frac{1}{4}(x^2+y^2-2)$ and $b(x,y) = 3x^2+3y^2-4$ are satisfactory.

7.* Solve the following equation! You can solve it by transforming it to canonical form - see Section 2.3. in the lecture notes!

$$\begin{cases} \partial_x^2 u(x,y) - \partial_y^2 u(x,y) - 2\partial_x u(x,y) + u(x,y) = y, \\ u(x,0) = e^x, \\ \partial_y u(x,0) = e^x + 1. \end{cases} \tag{2}$$

Solution: This can be submitted.