Fourth practice

Distributions I.: Definition, order

1. Let $\Omega \subset \mathbb{R}^n$ be an open and connected subset, and $\phi \in \mathcal{D}(\Omega)$ (where $\mathcal{D}(\Omega)$ is the set of test functions, as defined on the lecture).

   a) Let $\phi_j(x) := \frac{1}{j}\phi(x)$ $(x \in \Omega, j \in \mathbb{Z}^+)$. Show that $\phi_j(x) \xrightarrow{\mathcal{D}(\Omega)} 0$!

   Solution: Here $\text{supp}(\phi_j) = \text{supp}\phi$ for all j (since the support does not change if I divide my function by some number (the non-zero values remain non-zero, and the zero ones also stay zero). Also, $\forall\ \alpha$ multiindex

   $$|\partial^\alpha \phi_j(x)| = \frac{1}{j}|\partial^\alpha \phi(x)| \leq \frac{1}{j}\max_{\Omega}|\partial^\alpha \phi| \to 0$$

   which means that $\partial^\alpha \phi_j \to 0$ uniformly on Ω, therefore $\phi_j(x) \xrightarrow{\mathcal{D}(\Omega)} 0$.

   b) Let $\Omega = \mathbb{R}^n$, and $\phi_j(x) := \frac{1}{j}\phi\left(\frac{x}{j}\right)$ $(x \in \Omega, j \in \mathbb{Z}^+)$. Is it true that this sequence is convergent in the $\mathcal{D}(\Omega)$-sense?

   Solution: The key observation here is the fact that as $j \to \infty$, the support of $\phi\left(\frac{x}{j}\right)$ gets bigger and bigger (since if $y \in \text{supp}(\phi(x))$, then $jy \in \text{supp}\Big(\phi\left(\frac{x}{j}\right)\Big)$). But because of this, there is no such compact set which contains all of the supports (since it should be unbounded), meaning that this sequence cannot converge.

   c) Let $\Omega = \mathbb{R}^n$, and $\phi_j(x) := \frac{1}{j}\phi\left(jx\right)$ $(x \in \Omega, j \in \mathbb{Z}^+)$. Is it true that this sequence is convergent in the $\mathcal{D}(\Omega)$-sense?

   Solution: Here the support of functions $\phi_j(x)$ shrinks and get smaller as $j \to \infty$, so there is such a $K \subset \Omega$ compact set that it contains all of them. Also,

   $$|\phi_j(x)| \leq \frac{1}{j}\max_{\mathbb{R}^n}|\phi| \to 0$$

   meaning that $\phi_j \to 0$ uniformly.
   However, if we take $|\alpha| = 1$, then:

   $$\max_{\mathbb{R}^n}|\partial^\alpha \phi_j| = \max_{\mathbb{R}^n}|\partial^\alpha \phi|$$

   which means that $\partial^\alpha \phi \to 0$ (which should be true in case of convergence since $\phi_j \to 0$, so $\partial^\alpha \phi_j \to \partial^\alpha 0$) holds only if $\max_{\mathbb{R}^n}|\partial^\alpha \phi| = 0$, therefore $\partial^\alpha \phi = 0$, so ϕ is a constant, but it should have compact support, so it is only convergent if $\phi \equiv 0$.

2. Let $\Omega \subset \mathbb{R}^n$ be an open, connected set and $f \in L^1_{loc}(\Omega)$.

   a) Let $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ be defined as

   $$T_f(\phi) := \int_\Omega f\phi.$$

   (This is the regular distribution.) Show that this is a 0-order distribution!

   Solution: It is easy to see that this is linear (by definition). We also need that it should be sequentially continuous, for which we will use the theorem proved on the Lecture part, meaning that we have to find an upper bound for the value of the functional using the derivatives of ϕ. Let us suppose that $\text{supp}(\phi) \subset K$, and then:

   $$|T_f(\phi)| = \left|\int_\Omega f\phi\right| = \left|\int_K f\phi\right| \leq \int_\Omega |f\phi| \leq \max_K|\phi|\left(\int_K |f|\right)$$

   Since $f \in L^1_{loc}(\Omega)$, we know that $\left(\int_K |f|\right) = c_K < \infty$, so the theorem can be applied, and we get a 0-order distribution.

b) Let $U_{f,\beta} : \mathcal{D}(\Omega) \to \mathbb{R}$ be defined as

$$U_{f,\beta}(\phi) := \int_\Omega f \partial^\beta \phi$$

where β is a given multiindex. Prove that $U_{f,\beta} \in \mathcal{D}'(\Omega)$ and it has finite order! Also, is it possible that for two, (in the L^1-sense) different f and g functions $U_{f,\beta}(\phi) = U_{g,\beta}(\phi)$ for all $\phi \in \mathcal{D}(\Omega)$?

Solution: It is clear that this functional is linear. Then for some $\phi \in \mathcal{D}(\Omega)$ for which $\text{supp}(\phi) \subset K$, we get:

$$|U_{f,\beta}(\phi)| = \left| \int_\Omega f \partial^\beta \phi \right| = \left| \int_K f \partial^\beta \phi \right| \leq \int_K |f \partial^\beta \phi| \leq \max_K |\partial^\beta \phi| \left(\int_K |f| \right)$$

Since $f \in L^1_{loc}(\Omega)$, we know that $\left(\int_K |f|\right) = c_K < \infty$, so the theorem can be applied, and we get an at most $|\beta|$-order distribution.

For the second part, let us choose $n = 1$, $\Omega = (a, b)$, $f \in L^1_{loc}(a, b)$, $\beta = 1$ and $c \in \mathbb{R}$ arbitrary. Then for every $\phi \in \mathcal{D}(a, b)$:

$$\int_a^b (f + c)\phi' = \int_a^b f\phi' + c \int_a^b \phi' = \int_a^b f\phi' + c(\phi(b) - \phi(a)) = \int_a^b f\phi'$$

in which we used that since ϕ has compact support, then it takes zero values on the boundary, meaning that $\phi(b) = \phi(a) = 0$. Consequently, $U_{f,\beta} = U_{f+c,\beta}$. A similar thing can be shown in the case $|\beta| \geq 1$.

Remark: Note that in the case of $|\beta| = 0$, we get a regular distribution, and by the theorem stated on the lecture, in that case $f = g$ a.e.

3. Let $a \in \mathbb{R}^n$, and the functional $\delta_a : \mathcal{D} \to \mathbb{R}$ be defined as

$$\delta_a(\phi) := \phi(a)$$

(This is the Dirac-delta distribution.)

a) Prove that this is a zero-order distribution!

Solution: It is easy to see that it is linear. Also, if we apply the theorem from the lecture:

$$|\delta_a(\phi)| = |\phi(a)| \leq \max_K |\phi|$$

in which we supposed that $\text{supp}(\phi) \subset K$. Then δ_a is a 0-order distribution.

Remark: In this case we do not even need the aforementioned theorem, but the sequentially continuous property can be proved directly, in the following way. Let us suppose that $\phi_j \xrightarrow{\mathcal{D}(\Omega)} \phi$. Then it means that $\phi_j \to \phi$ uniformly, from which we get that $\phi_j \to \phi$ pointwise also, meaning that $\phi_j(a) \to \phi(a)$, which gives our property.

b) Show that δ_a is not a regular distribution!

Solution: We prove the statement by contradiction: let us suppose that $\exists f \in L^1_{loc}(\Omega)$ such that $\phi(a) = \int_\Omega f\phi$. Since $f \in L^1_{loc}(\Omega)$, then there is a neighborhood of a denoted by U, for which $\int_U |f| < 1$. (Such an interval can be constructed by the approximation theorem.)

Let ϕ be such a test function for which $\text{supp}(\phi) \subset U$, and $\max |\phi| = \phi(a)$ (such a test function exists, see the construction of the η functions on the lecture). Then we get the following:

$$|\phi(a)| = \left| \int_\Omega f\phi \right| \leq \int_\Omega |f||\phi| = \int_U |f||\phi| \leq \max_U |\phi| \int_U |f| < |\phi(a)|$$

In the last step we used the definition of U and ϕ. This is clearly a contradiction, so we get our statement.

Remark: The above statement can be also proved if we observe the support of the two functionals. We know that $\text{supp}(\delta_a) = \{a\}$, and $\text{supp}(T_f) = \text{supp}(f)$, meaning that these two can only be equivalent if we have an L^1_{loc} function which has a support of just point a, but then $f = 0$ a.e., meaning that $\delta_a(\phi) = T_f(\phi) = T_0(\phi) = 0$ which is clearly a contradiction.

4. Let $\Omega = (0,2)$ and $u : \mathcal{D}(\Omega) \to \mathbb{R}$ defined as

$$u(\phi) := \sum_{j=1}^{\infty} \phi^{(j)}\left(\frac{1}{j}\right).$$

a) Show that $u \in \mathcal{D}'(\Omega)$!

Solution: Let $K \subset (0{,}2)$ be a compact set. Then since K has a smallest value, then there is such an $N \in \mathbb{Z}^+$ value for which $\frac{1}{N} \in K$, but $\frac{1}{N+k} \notin K$ for all $k \in \mathbb{Z}^+$. Then if we have a function ϕ such that $\operatorname{supp}(\phi) \subset K$, then $\phi\left(\frac{1}{N+k}\right) = 0$ for all $k \in \mathbb{Z}^+$, meaning that

$$|u(\phi)| = \left|\sum_{j=1}^{\infty} \phi^{(j)}\left(\frac{1}{j}\right)\right| = \left|\sum_{j=1}^{N} \phi^{(j)}\left(\frac{1}{j}\right)\right| \leq \sum_{j=1}^{N} \max_K |\phi^{(j)}|$$

which means that by our theorem, u is sequentially continuous, and since it is also linear, then u is a distribution.

b) * Does u have finite order?

Solution: The solution can be submitted.