Fifth practice

Distributions II.: Derivation

1. Let $f \in C^m(\mathbb{R}^n)$. Show that for any $|\alpha| \leq m$ multiindex, $\forall \phi \in C_0^\infty(\mathbb{R}^n)$,
$$T_{\partial^\alpha f}(\phi) = (-1)^{|\alpha|}\, T_f(\partial^\alpha \phi).$$
(This was stated on the Lecture, but was not proved there.)

   Solution: The more general statement can be proved by induction - here we only prove the special case $|\alpha| = 1$ (so there is only one, first order derivative):
$$T_{\partial_j f}(\phi) = -T_f(\partial_j \phi).$$
Without loss of generality let us assume that $j = 1$ (the derivative is in the first variable), and let us use the notation for $x \in \mathbb{R}^n$: $x = (x_1, \tilde{x})$, where $\tilde{x} \in \mathbb{R}^{n-1}$ (so here x_1 is the first variable, and $\tilde{x}$ contains all the other, $n-1$-many variables).

   Then for $f \in C^1(\mathbb{R}^n)$, f and $\partial_1 f$ are in $L^1_{loc}(\mathbb{R}^n)$, which means that the notations T_f and $T_{\partial_1 f}$ make sense. Then by definition:
$$T_{\partial_1 f}(\phi) = \int_{\mathbb{R}^n} (\partial_1 f)\phi = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_1 f(x_1, \tilde{x})\phi(x_1, \tilde{x}) dx_1 d\tilde{x} =$$
Now we apply integration by parts to the inner integral:
$$= \int_{\mathbb{R}^{n-1}} \left[f(x_1, \tilde{x})\phi(x_1, \tilde{x})\right]_{x_1=-\infty}^{\infty} d\tilde{x} - \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x_1, \tilde{x})\partial_1 \phi(x_1, \tilde{x}) dx_1 d\tilde{x} =$$
Now we use the fact that ϕ has compact support, so ϕ has zero values at $-\infty$ and at ∞, meaning that the first term is zero.
$$= -\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x_1, \tilde{x})\partial_1 \phi(x_1, \tilde{x}) dx_1 d\tilde{x} = -\int_{\mathbb{R}^n} f \partial_1 \phi = -T_f(\partial_1 \phi),$$
which gives our statement.

2. Let $\partial^\alpha u(\phi) := (-1)^{|\alpha|} u(\partial^\alpha \phi)$. Show that $\partial^\alpha u$ is a distribution!

   (This proposition was also stated at the Lecture, but is proved here.)

   Solution: Here we only prove the statement for $|\alpha| = 1$, namely that $\partial_j u(\phi) := -u(\partial_j \phi)$ is indeed a distribution. The more general case can be done by induction.

   For $\partial_j u$ to be a distribution, we need two properties: it should be linear, and also sequentially continuous. The linearity is trivially true. For the sequentially continuous property, let us assume that $\phi_k \xrightarrow{\mathcal{D}(\Omega)} \phi$. This means that then $\partial_j \phi_k \to \partial_j \phi$, and this also holds for all the derivatives of ϕ_k and ϕ; also, the supports are inside a compact set (because of the definition of $\phi_k \xrightarrow{\mathcal{D}(\Omega)} \phi$). So this means that $\partial_j \phi_k \xrightarrow{\mathcal{D}(\Omega)} \partial_j \phi$ also holds, and since u is a distribution, it is sequentially continuous, so $u(\partial_j \phi_k) \to u(\partial_j \phi)$ is also true, meaning that $u(\partial_j \phi)$ is sequentially continuous, so it is a distribution.

3. Let $a \in \mathbb{R}^n$. What is $\partial^\alpha \delta_a(\phi)$?

   Solution: By the definition of derivation and the definition of δ_a,
$$\partial^\alpha \delta_a(\phi) = (-1)^{|\alpha|}\delta_a(\partial^\alpha \phi) = (-1)^{|\alpha|}\partial^\alpha(\phi(a)).$$

4. Prove the following statements!

   a) $T'_{|.|} = T_{\text{sgn}}$, (where $|.|$ is the absolute value function, and sgn is the signum (or sign) function).

   Solution: Let $\phi \in \mathcal{D}(\mathbb{R})$ be a arbitrary function. Then by definition:

$$T'_{|.|}(\phi) = -T_{|.|}(\phi') = -\int_{-\infty}^{\infty} |x|\phi'(x)dx =$$

   Now we split the integral into two parts (because $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$):

$$= -\int_{0}^{\infty} x\phi'(x)dx + \int_{-\infty}^{0} x\phi'(x)dx =$$

   Now we use integration by parts:

$$= -[x\phi(x)]_{x=0}^{\infty} + \int_{0}^{\infty} \phi(x)dx + [x\phi(x)]_{-\infty}^{0} - \int_{-\infty}^{0} \phi(x)dx =$$

   Now we use the fact that $\phi(x)$ has compact support, so its values at ∞ and at $-\infty$ are zero (also, $x = 0$ at zero).

$$= \int_{0}^{\infty} \phi(x)dx - \int_{-\infty}^{0} \phi(x)dx = \int_{0}^{\infty} 1 \cdot \phi(x)dx + \int_{-\infty}^{0} (-1) \cdot \phi(x)dx =$$

$$= \int_{-\infty}^{\infty} \text{sgn}(x)\phi(x)dx = T_{\text{sgn}}(\phi).$$

   b) $T'_{\text{sgn}} = 2\delta_0$.

   Solution: By definition:

$$T'_{\text{sgn}}(\phi) = -T_{\text{sgn}}(\phi') = -\int_{-\infty}^{\infty} \text{sgn}(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx + \int_{-\infty}^{0} \phi'(x)dx =$$

   Now by the fundamental theorem of calculus (Newton-Leibniz formula):

$$= -[\phi(x)]_0^{\infty} + [\phi(x)]_{-\infty}^{0} =$$

   We know that ϕ has compact support, so its values at ∞ and at $-\infty$ are zero:

$$= 2\phi(0) = 2\delta_0(\phi).$$

   c) $T'_H = \delta_0$. (where H is the Heaviside function, which takes 0 for negative, and 1 for non-negative values)

   Solution: By definition:

$$T'_H(\phi) = -T_H(\phi') = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx$$

   Now by the fundamental theorem of calculus (the Newton-Leibniz formula):

$$= -[\phi(x)]_0^{\infty} = \phi(0) = \delta_0(\phi),$$

   where we used that ϕ has compact support, so its values at ∞ are zero.

   Remark: All the previous statements are also easy corollaries of the proposition stated on the lecture about the derivative of piece-wise differentiable functions.

5. Is there such a distribution for which $u' = \delta_{-1} + \delta_1$?

   Solution:

   First method: We are using the proposition stated on the lecture about the derivative of piece-wise differentiable functions, namely, that if $f \in C^1(a_i, a_{i+1})$ $\forall i$, and $\exists \lim_{a_i+} f(x)$ and $\exists \lim_{a_i-} f(x)$ for all i, then

$$T'_f = T_{f'} + \sum_i \left(\lim_{a_i+} f(x) - \lim_{a_i-} f(x) \right) \delta_{a_i}$$

(Here $\lim_{a_i+} f(x) - \lim_{a_i-} f(x)$ is the jump of the function f at point a_i.)

So if we choose such a function which has zero derivative for every point inside our small intervals, and it has a jump at $x = -1$ and one at $x = 1$, then we are fine. So let

$$f(x) = \begin{cases} 0, & \text{if } x < -1, \\ 1, & \text{if } -1 \leq x < 1, \\ 2, & \text{if } x \geq 1. \end{cases}$$

Second method: By exercise 4. c), if we have a Heaviside function with jump at -1 (denoted by H_{-1}), then $T'_{H_{-1}} = \delta_{-1}$. Similarly, for a Heaviside with jump at 1, we get $T'_{H_1} = \delta_1$. This means that if $u = T_{H_{-1}} + T_{H_1}$, then $u' = T'_{H_{-1}} + T'_{H_1} = \delta_{-1} + \delta_1$.

6. Show that the distribution $u(x) = H(x)\sin(x)$ is a solution of the following differential equation (in the distribution sense):

$$u'' + u = \delta_0.$$

Solution: Our goal here is to prove that $T''_u(\phi) + T_u(\phi) = \delta_0(\phi)$, or in other words, $T''_u(\phi) = \delta_0(\phi) - T_u(\phi)$. By definition:

$$T''_u(\phi) = \int_{-\infty}^{\infty} H(x)\sin(x)\phi''(x)dx = \int_0^{\infty} \sin(x)\phi''(x)dx =$$

If we use integration by parts:

$$= [\sin(x)\phi'(x)]_0^{\infty} - \int_0^{\infty} \cos(x)\phi'(x)dx =$$

Now we use the fact that $\phi(x)$ has compact support, so its derivative at ∞ is zero, and $\sin(0) = 0$, so $[\sin(x)\phi'(x)]_0^{\infty} = 0$.

$$= -\int_0^{\infty} \cos(x)\phi'(x)dx =$$

If we use integration by parts:

$$= -[\cos(x)\phi(x)]_0^{\infty} - \int_0^{\infty} \sin(x)\phi(x)dx =$$

Now we use the fact that $\phi(x)$ has compact support, so its value at ∞ is zero, and $\cos(0) = 1$, so $-[\cos(x)\phi(x)]_0^{\infty} = \phi(0)$.

$$= \phi(0) - \int_0^{\infty} \sin(x)\phi(x)dx = \phi(0) - \int_{-\infty}^{\infty} H(x)\sin(x)\phi(x)dx = \delta_0(\phi) - T_u(\phi).$$

7. Let us define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \begin{cases} \frac{1}{2} & \text{if } xy \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is $\partial_{12}T_f$ (where ∂_{12} means the mixed derivative)?

Solution: By definition,

$$\partial_{12}T_f(\phi) = (-1)^2 T_f(\partial_{12}\phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\ \partial_{12}\phi =$$

By the definition of f, it takes the values $\frac{1}{2}$ only if x and y have the same sign, which means that:

$$= \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \partial_{12}\phi\ dxdy + \frac{1}{2}\int_0^{\infty}\int_0^{\infty} \partial_{12}\phi\ dxdy =$$

From the fundamental theorem of calculus (the Newton-Leibniz formula), and from the fact that ϕ has compact support:

$$= \frac{1}{2}\phi(0{,}0) + \frac{1}{2}\phi(0{,}0) = \phi(0{,}0) = \delta_{(0,0)}(\phi).$$

8. Let us define the multi-variable Heaviside function $\tilde{H} : \mathbb{R}^n \to \mathbb{R}$ as

$$\tilde{H}(x) = \begin{cases} 1 & \text{if } x_i \geq 0 \ \forall i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

a) Show that $\partial_1 \partial_2 \ldots \partial_n \tilde{H} = \delta_{\mathbf{0}}$.
Solution: By definition:

$$\partial_1 \partial_2 \ldots \partial_n \tilde{H}(\phi) = (-1)^n \int_{\mathbb{R}^n} \tilde{H}(\mathbf{x}) \partial_1 \partial_2 \ldots \partial_n \phi(\mathbf{x}) d\mathbf{x} =$$

$$= (-1)^n \int_0^\infty \int_0^\infty \cdots \int_0^\infty \partial_1 \partial_2 \ldots \partial_n \phi(\mathbf{x}) dx_1 dx_2 \ldots dx_n =$$

Now apply the fundamental theorem of calculus (Newton-Leibniz rule) in variable x_1:

$$= (-1)^n \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left[\partial_2 \partial_3 \ldots \partial_n \phi(x_1, x_2, \ldots x_n)\right]_{x_1=0}^\infty dx_2 \ldots dx_n =$$

Now we use the fact that ϕ has compact support:

$$= (-1)^{n-1} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \partial_2 \partial_3 \ldots \partial_n \phi(0, x_2, \ldots x_n) dx_2 \ldots dx_n =$$

By continuing this process, we get:

$$= (-1)^{n-2} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \partial_3 \ldots \partial_n \phi(0, 0, x_3, \ldots x_n) dx_3 \ldots dx_n =$$

$$= \cdots = (-1)^0 \phi(0{,}0, \ldots 0) = \delta_{\mathbf{0}}(\phi).$$

in which $\mathbf{0}$ is the all-zero vector.

b) Let us define the function $r(x) : \mathbb{R}^n \to \mathbb{R}$ as

$$r(x) = \begin{cases} x_1 x_2 \ldots x_n, & \text{if } x_i \geq 0 \ \forall i, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then show that $\partial_1 \partial_2 \ldots \partial_n T_r = \tilde{H}$.
Solution: By definition:

$$\partial_1 \partial_2 \ldots \partial_n T_r(\phi) = (-1)^n \int_{\mathbb{R}^n} r(\mathbf{x}) \ \partial_1 \partial_2 \ldots \partial_n \phi(\mathbf{x}) \ d\mathbf{x} =$$

$$= (-1)^n \int_0^\infty \int_0^\infty \cdots \int_0^\infty x_1 x_2 \ldots x_n \ \partial_1 \partial_2 \ldots \partial_n \phi(\mathbf{x}) \ dx_1 dx_2 \ldots dx_n =$$

Now use integration by parts in variable x_1:

$$= (-1)^n \int_0^\infty \cdots \int_0^\infty \left(\left[x_1 \ldots x_n \partial_2 \ldots \partial_n \phi(\mathbf{x})\right]_{x_1=0}^\infty - \int_0^\infty x_2 \ldots x_n \partial_2 \ldots \partial_n \phi(\mathbf{x}) dx_1 \right) dx_2 \ldots dx_n =$$

The first term is 0 at $x_1 = 0$, and it is also zero at ∞ because ϕ has compact support, so:

$$= (-1)^{n-1} \int_0^\infty \cdots \int_0^\infty x_2 \ldots x_n \ \partial_2 \ldots \partial_n \phi(\mathbf{x}) dx_1 dx_2 \ldots dx_n =$$

By continuing this same process, we get:

$$= (-1)^{n-2} \int_0^\infty \cdots \int_0^\infty x_3 \ldots x_n \ \partial_3 \ldots \partial_n \phi(\mathbf{x}) dx_1 dx_2 \ldots dx_n =$$

$$= \cdots = (-1)^0 \int_0^\infty \cdots \int_0^\infty \phi(\mathbf{x}) dx_1 dx_2 \ldots dx_n = \int_{\mathbb{R}^n} \tilde{H}(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x} = T_{\tilde{H}}(\phi).$$

Remark: The previous statements can also be proved using a more general form of the proposition which was stated on the lecture about piece-wise differentiable functions.

9. Let us define the following functional:

$$u(\phi) := \int_0^\infty \phi(0,y)dy.$$

a) Show that $u \in \mathcal{D}'(\mathbb{R}^2)$!

Solution: It is easy to see that it is indeed linear. For the sequentially continuous property we are going to use the theorem proved on the Lecture. For this, let us define a compact set $K \subset \mathbb{R}^2$, and consider a test function $\phi \in \mathcal{D}(\mathbb{R}^2)$ for which $\text{supp}(\phi) \subset K$. Then there is such an $R > 0$ constant, for which $K \subset B(0,R)$ (since K is bounded). Then (by transforming to polar coordinates):

$$|u(\phi)| \leq \int_0^\infty |\phi(0,y)|dy \leq \int_0^R \int_0^{2\pi} \max_K |\phi| \; r \; d\theta \; dr \leq \text{const} \cdot \max_K |\phi|,$$

which means that u is a zero-order distribution.

b) Show that $\partial_2 u = \delta_{(0,0)}$.

Solution: By definition:

$$\partial_2 u(\phi) = -u(\partial_2 \phi) = -\int_0^\infty \partial_2\phi(0,y)dy = \phi(0{,}0) = \delta_{(0,0)}(\phi),$$

in which we used that ϕ has compact support.

c) Show that there is such an $f \in L^1_{loc}(\mathbb{R}^2)$ function, for which $u = \partial_1 T_f$.

Solution: First let us consider the term inside the definition of u:

$$\phi(0,y) = -\left[\phi(x,y)\right]_{x=0}^\infty = -\int_0^\infty \partial_x\phi(x,y)dx = -\int_{-\infty}^\infty H(x)\partial_x\phi(x,y)dx.$$

in which we used that ϕ has compact support, the fundamental theorem of calculus (the Newton-Leibniz formula) and the definition of $H(x)$. Now let us consider $u(\phi)$:

$$u(\phi) = \int_0^\infty \phi(0,y)dy = \int_{-\infty}^\infty H(y)\phi(0,y)dy =$$

Now we substitute into the previous form:

$$= -\int_{-\infty}^\infty H(y)\int_{-\infty}^\infty H(x)\partial_x\phi(x,y)dxdy = -\int_{-\infty}^\infty\int_{-\infty}^\infty H(y)H(x)\partial_x\phi(x,y)dxdy =$$

Let us define f as $f(x,y) := H(x)H(y)$, then:

$$= -\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y)\partial_x\phi(x,y)dxdy = -T_f(\partial_x\phi(x,y)) = \partial_x T_f(\phi(x,y))$$

So there is such an $f \in L^1_{loc}(\mathbb{R}^2)$ function for which $u = \partial_1 T_f$ holds.

10.* Let $u \in \mathcal{D}'(\Omega)$ in a way that $u' = 0$. Is it true then that $u = c$ for some constant $c \in \mathbb{R}$?

Solution: This can be submitted.