Seventh practice

Parabolic equations

1. Let $a>0$ and $b\in\mathbb{R}$ be constants. Prove that
$$\int_{-\infty}^{+\infty} e^{-ay^2}\cos by\,dy=\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}.$$

Solution: First we prove that for all $a>0$ and $b\in\mathbb{R}$ values the above integral exists and it is finite. Indeed, since $|\cos by|\le 1$, then the integrant has an integrable majorant (namely e^{-ay^2}). Also observe that in the special case of $b=0$ we can compute the value of the integral easily, since (using $x=ay$)
$$\int_{-\infty}^{+\infty} e^{-ay^2}\,dy=\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} e^{-x^2}\,dx=\sqrt{\frac{\pi}{a}}.$$

To compute our initial integral, let us consider it as a function of $a\in\mathbb{R}^+$ and $b\in\mathbb{R}$, and define the following function $I\colon\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$:
$$I(a,b)=\int_{-\infty}^{+\infty} e^{-ay^2}\cos by\,dy.$$

Differentiate I by its second variable! Then by the theory of parametric integrals, if the derivative of the integrand (taken in the second variable) has an integrable majorant (not depending on the parameter), then the differentiation can be done in a way that we differentiate only the integrand. By simple computations we get that $\partial_b e^{-ay^2}\cos by=-ye^{-ay^2}\sin by$. Let us restrict the domain of function I to the set $[a_0,+\infty)\times\mathbb{R}$, in which $a_0>$ is fixed. Then in the case $a>a_0$ we get $\left|-ye^{-ay^2}\sin by\right|\le|y|e^{-a_0y^2}$, which has a convergent improper integral (it can even be computed), so the theory mentioned above can be applied. Consequently, (in which we integrate by parts in variable y):
$$\partial_b I(a,b)=\int_{-\infty}^{+\infty} -ye^{-ay^2}\sin by\,dy=\frac{1}{2a}\int_{-\infty}^{+\infty} -2aye^{-ay^2}\sin by\,dy=$$
$$=\frac{1}{2a}\left(\left[e^{-ay^2}\sin by\right]_{y=-\infty}^{+\infty}-b\int_{-\infty}^{+\infty} e^{-ay^2}\cos by\,dy\right)=-\frac{b}{2a}I(a,b).$$

This means that function I satisfies the following initial-value problem:
$$\frac{\partial_b I(a,b)}{I(a,b)}=-\frac{b}{2a},\quad I(a,0)=\sqrt{\frac{\pi}{a}}.$$

Now by integrating both sides of the above ordinary differential equation in variable b, we get that $\log I(a,b)=-\frac{b^2}{4a}+c(a)$, meaning that $I=C(a)\exp\left(-\frac{b^2}{4a}\right)$, in which $C(a)$ is a constant depending on a. Then by the substitution $b=0$, and using the initial value $I(a,0)$ we get
$$I(a,b)=\sqrt{\frac{\pi}{a}}\exp\left(-\tfrac{b^2}{4a}\right).$$

Note that the above reasoning works for all $a>a_0>0$ values, so it is true for any $a>0$, which means that we proved the statement.

Remark: The exercise could have been solved by the following observation:
$$\int_{-\infty}^{+\infty} e^{-ay^2}\cos by\,dy=\mathrm{Re}\int_{-\infty}^{+\infty} e^{-ay^2}e^{\mathrm{i}by}\,dy=e^{-\frac{b^2}{4a}}\mathrm{Re}\int_{-\infty}^{+\infty} e^{-a\left(y-\mathrm{i}\frac{b}{2}\right)^2}\,dy=\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}},$$

in which we used that

$$\int_{-\infty}^{+\infty} e^{-az^2}\, dy = \sqrt{\frac{\pi}{a}}$$

for all $z \in \mathbb{C}$, since it is true for all $z \in \mathbb{R}$, and then by the unicity theorem it should be also true for all complex values of z.

2. Suppose that $g \in C(\mathbb{R}^n)$ is bounded, and let $u(t,x) = \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} g(x - 2\sqrt{t}\eta)\, d\eta$.

   a) Prove that $u(0,x) = g(x)$ holds for all $x \in \mathbb{R}^n$.

   b) Assume that $g \in C^2(\mathbb{R}^n)$, for which $g, \partial_j g, \partial_j^2 g$ $(j = 1, \dots, n)$ are bounded. Show that then $\partial_t u - \Delta u = 0$ inside $\mathbb{R}^+ \times \mathbb{R}^n$.

Solution: a) First we note that since g is bounded, the integral exists and is finite. Also, by the continuity of parametric improper integrals

$$u(0,x) = \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} g(x)\, d\eta = g(x) \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2}\, d\eta = g(x),$$

in which we used that $\int_{\mathbb{R}^n} e^{-|\eta|^2}\, d\eta = (\sqrt{\pi}\,)^n$ (see Exercise 3 from Practice 6).

b) Because of the assmupltions, the differentiation can be moved inside the integral (by the differentiability of parametric improper integrals). Then by simple calculations we get that

$$\partial_t u(t,x) = -\sum_{j=1}^{n} \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} \partial_j g(x - 2\sqrt{t}\eta) \frac{\eta_j}{\sqrt{t}}\, d\eta_j$$

and

$$\begin{aligned} \partial_{x_j}^2 u(t,x) &= \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} \partial_j^2 g(x - 2\sqrt{t}\eta)\, d\eta = \\ &= \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^{n-1}} \left(\left[-\frac{1}{2\sqrt{t}} e^{-|\eta|^2} \partial_j g(x - 2\sqrt{t}\eta) \right]_{\eta_j = -\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-|\eta|^2} \partial_j g(x - 2\sqrt{t}\eta) \frac{\eta_j}{\sqrt{t}}\, d\eta_j \right), \end{aligned}$$

in which we used a partial integral. Observe that the function inside the brackets $[\dots]$ tends to zero if $|\eta| \to \infty$, because $\partial_j g$ is bounded, and all the other terms tend to zero, when $|\eta| \to \infty$. Consequently,

$$\Delta u(t,x) = -\sum_{j=1}^{n} \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} \partial_j g(x - 2\sqrt{t}\eta) \frac{\eta_j}{\sqrt{t}}\, d\eta_j$$

so $\partial_t u = \Delta u$ holds inside $\mathbb{R}^+ \times \mathbb{R}^n$.

It is important to realize that all the above calculations hold if the derivatives of function g do not "grow too fast" (e.g. the magnitude of their growth is $e^{c|x|}$).

Note that we could have solved the problem in a different way. By the transformation $\xi = x - 2\sqrt{t}\eta$ we get that

$$\frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} g(x - 2\sqrt{t}\eta)\, d\eta = \int_{\mathbb{R}^n} \frac{1}{(2\sqrt{\pi t}\,)^n} e^{-\frac{|x-\xi|^2}{4t}} g(\xi)\, d\xi.$$

We proved in Exercise 4 on Practice 6 that the function $(t,x) \mapsto \frac{1}{(2\sqrt{\pi t}\,)^n} e^{-\frac{|x-\xi|^2}{4t}}$ (more precisely, its dilatation) is the solution of the heat equation, so by the assumptions its integral is also one. In this case it is enough to assume that $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we don't need differentiability. Note that the solution u is always inside $C^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ (it is even analytic), and this does not depend on the smoothness of g. This property of the heat equation is sometimes called *parabolic smoothing.*

Another important remark is that the Cauchy problem of the heat equation has infinitely many solutions, the above formula only gives us one of them. Thikhonov (1906–1993) gave an easy construction for such infinite set of solutions, these solutions "grow fast" as $|x| \to \infty$ (with magnitude $e^{|x|^\alpha}$). If we assume a slower growth, then the solution is unique, and the above formula holds. Also, David Vernon Widder (1898–1990) showed that even if we have infinitely many solutions, there is only one, which is non-negative, and this is the interesting one, if u means absolute temperature.

From now on, when we talk about "the solution of the Cauchy-problem", we think of the solution described by the above formula.

3. Solve the following Cauchy-problems!

a) $\begin{cases} \partial_t u - \partial_x^2 u &= 0 \quad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) &= x \quad (x \in \mathbb{R}). \end{cases}$

b) $\begin{cases} \partial_t u - \partial_x^2 u &= 0 \quad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) &= \cos x \quad (x \in \mathbb{R}). \end{cases}$

Solution: We use the formula which we proved in Exercise 2.

a)

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2}(x - 2\sqrt{t}\eta)\, d\eta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} x\, d\eta - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} 2t\eta\, d\eta = x,$$

using that $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^n} e^{-|\eta|^2}\, d\eta = 1$, and the second integrand is an odd function, so its integral is zero on an interval which is symmetric to the origin.

b) By the formula,

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} \cos(x - 2\sqrt{t}\eta)\, d\eta.$$

Now we apply the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, then we get

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} \cos(x - 2\sqrt{t}\eta)\, d\eta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} \left(\cos x \cos 2\sqrt{t}\eta + \sin x \sin 2\sqrt{t}\eta\right) d\eta =$$
$$= \cos x \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} \cos 2\sqrt{t}\eta\, d\eta = e^{-t} \cos x,$$

in which we used that sin is an odd function, so its integral is zero on an interval symmetric to the origin, and by Exercise 1 we know that

$$\int_{-\infty}^{\infty} e^{-ay^2} \cos by\, dy = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}},$$

so we can use it with the choices $a = 1$ and $b = 2\sqrt{t}$. In conclusion, $u(t,x) = e^{-t}\cos x$.

4. Let $n = 1$, $f = 0$, and observe the parabolic Cauchy-problem.

   a) Assume that $g \in C(\mathbb{R})$ is bounded. Prove that if $g(x) \geq 0$ $(x \in \mathbb{R})$, then for the solution u of the Cauchy-problem $u(x) \geq 0$ $(x \in \mathbb{R})$.

   b) Assume that $g \in C^2(\mathbb{R})$ and g, g', g'' are bounded. Prove that if g is convex, then for all $t > 0$ the solution u of the Cauchy-problem $u(t, \cdot)$ is also convex.

Solution: a) If $g(x) \geq 0$ for all $x \in \mathbb{R}$, then

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} g(x - 2\sqrt{t}\eta)\, d\eta \geq 0,$$

since the integral of a non-negative function is non-negative.

b) If g is convex, then $g''(x) \geq 0$ for all $x \in \mathbb{R}$, so

$$\partial_x^2 u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2} g''(x - 2\sqrt{t}\eta)\, d\eta \geq 0,$$

meaning that $u(t, \cdot)$ is a convex function.

5. Prove that the solution of the parabolic Cauchy-problem depends continuously on g in the following sense: if $g_1,\ g_2 \in C(\mathbb{R}^n)$ are bounded, for which

$$|g_1(x) - g_2(x)| \le \varepsilon\ (x \in \mathbb{R}^n),$$

then for the corresponding solutions $u_1,\ u_2$ of the Cauchy-problem

$$|u_1(t,x) - u_2(t,x)| \le \varepsilon\ \ ((t,x) \in \mathbb{R}_0^+ \times \mathbb{R}).$$

Solution: By the assumptions, we get from the formula that

$$|u_1(t,x) - u_2(t,x)| \le \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} |g_1(x) - g_2(x)|\, d\eta \le \frac{\varepsilon}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2}\, d\eta = \varepsilon.$$

6. Let $n = 1$, $f = 0$, $g \in C(\mathbb{R})$, and assume that $\operatorname{supp} g \subset [a,b]$, and $g|_{[a,b]} > 0$. Prove that then for the solution u of the parabolic Cauchy-problem $u(t,x) > 0$ for all $(t,x) \in \mathbb{R}^+ + \times \mathbb{R}$! (Heat moves with infinite speed.)

Solution: From the conditions $g(x - 2\sqrt{t}\eta) > 0$, if $\eta \in I = \left[\frac{x-b}{2\sqrt{t}}, \frac{x-a}{2\sqrt{t}}\right]$, and for η-s outside of this interval $g(x - 2\sqrt{t}\eta) = 0$. Then

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_I e^{-|\eta|^2} g(x - 2\sqrt{t}\eta)\, d\eta > 0,$$

since the integral of a positive function is positive.

7.* Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ continuous, for which $\partial_1 g$ exists ad it is continuous on $\mathbb{R}^2$. Let us define function $f\colon \mathbb{R} \to \mathbb{R}$ in a way that $f(x) = \int_a^x g(x,y)\, dy$, in which $a \in \mathbb{R}$ is fixed. Prove that $f'(x) = g(x,x) + \int_a^x \partial_1 g(x,y)\, dy$.

Solution: The solution can be submitted.

8. Consider the following set of problems:

$$\begin{cases} \partial_t v - \Delta v &= 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ v(0,x) &= f(\tau,x) & (x \in \mathbb{R}^n) \end{cases}, \tag{1}$$

in which $\tau \in \mathbb{R}_0^+$ is a parameter. Suppose that for all $\tau \in \mathbb{R}_0^+$, for the solutions $v(\cdot,\cdot;\tau)$ of the equation, $v, \partial_t v,\ \Delta v \in C(\mathbb{R}_0^+ \times \mathbb{R}^n \times \mathbb{R}_0^+)$ holds. Define function u as:

$$u(t,x) = \int_0^t v(t-\tau, x; \tau) d\tau.$$

Prove that $\partial_t u - \Delta u = f$ inside $\mathbb{R}^+ \times \mathbb{R}^n$ and $u(0,x) = 0$ $(x \in \mathbb{R}^n)$, so u is a solution of the second sub-problem. (Duhamel-principle)

Solution: It is clear that $u(0,x) = \int_0^0 (\dots) = 0$ $(x \in \mathbb{R}^n)$. Then by applying the rule of derivation from Exercise 7:

$$\partial_t u(t,x) = v(0,x;t) + \int_0^t \partial_t v(t-\tau,x;\tau)\, d\tau = f(t,x) + \int_0^t \partial_t v(t-\tau,x;\tau)\, d\tau,$$

since $v(\cdot,\cdot;\tau)$ is the solution of problem (1) with parameter $\tau = t$, so $v(0,x;t) = f(t,x)$. Moreover, $\Delta u(t,x) = \int_0^t \Delta v(t-\tau,x;\tau)\, d\tau$, since by the conditions, the derivative of the integral is the integral of the derivative of the ingerand. Therefore,

$$\partial_t u(t,x) - \Delta u(t,x) = f(t,x) + \int_0^t \left(\partial_t v(t-\tau,x;\tau) - \Delta v(t-\tau,x;\tau)\right) d\tau = f(t,x),$$

since $\partial_t v(t-\tau,x;\tau) - \Delta v(t-\tau,x;\tau) = 0$ holds for all $\tau \in \mathbb{R}^+$, because $v(\cdot,\cdot;\tau)$ is the solution of problem (1).

Remark: Note that by Exercise 2, we can even compute functions $v(\cdot, \cdot; \tau)$, namely

$$v(t, x; \tau) = u(t, x) = \frac{1}{(\sqrt{\pi}\,)^n} \int_{\mathbb{R}^n} e^{-|\eta|^2} f(\tau, x - 2\sqrt{t}\eta)\, d\eta = \frac{1}{(2\sqrt{\pi t}\,)^n} \int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4t}} f(\tau, \xi)\, d\xi,$$

in which we used the substitution $\xi = x - 2\sqrt{t}\eta$ (and we used that the Jacobian is $\frac{1}{(2\sqrt{t}\,)^n}$). Then the solution of the second sub-problem is

$$u_2(t, x) = \int_0^t \frac{1}{(2\sqrt{\pi(t-\tau)}\,)^n} \int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} f(\tau, \xi)\, d\xi\, d\tau.$$

So the solution of the parablic Cauchy-problem

$$\begin{cases} \partial_t v - \Delta v &= f & \mathbb{R}^+ \times \mathbb{R}^n\text{-ben}, \\ v(0, x) &= g(x) & (x \in \mathbb{R}^n) \end{cases}$$

has the form

$$u(t, x) = \int_0^t \frac{1}{(2\sqrt{\pi(t-\tau)}\,)^n} \int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} f(\tau, \xi)\, d\xi\, d\tau + \frac{1}{(2\sqrt{\pi t}\,)^n} \int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4t}} g(\xi)\, d\xi.$$

Remark: The Duhamel-principle holds for more general equations in the form $\partial_t u - Lu = f$, in which L is a differential operator with constant coefficients. The principle can even extended to hyperbolic probelms, see the next Sheet of exercises. It even holds for ordinary differential equations: the solution of the initial-value problem

$$\begin{aligned} y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_1 y' + a_0 y &= f, \\ y^{(j)}(0) &= 0 \qquad (j = 0, \dots, n-1) \end{aligned}$$

can be computed from the solution y_τ of the problem

$$\begin{aligned} y_\tau^{(n)} + a_{n-1} y_\tau^{(n-1)} + \dots a_1 y_\tau' + a_0 y_\tau &= 0, \\ y_\tau^{(j)}(0) &= 0 \qquad (j = 1, \dots, n-1) \\ y_\tau(0) &= f(\tau) \end{aligned}$$

by the integral

$$y(t) = \int_0^t y_\tau(t - \tau)\, d\tau.$$

In other words,

$$y(t) = \int_0^t f(\tau) \tilde{y}(t - \tau)\, d\tau,$$

in which $\tilde{y}$ is the solution of the initial-value problem

$$\begin{aligned} \tilde{y}^{(n)} + a_{n-1} \tilde{y}^{(n-1)} + \dots a_1 \tilde{y}' + a_0 \tilde{y} &= f \\ \tilde{y}^{(j)}(0) &= 0 \qquad (j = 0, \dots, n-1), \end{aligned}$$

so $\tilde{y}$ (by extending it to the negative values as zero) is a fundamental solution of the differential-operator (and then the solution y can be computed as the convolution of the fundamental solution and the right-had side).

8. Solve the following Cauchy-problems!

a) $$\begin{cases} \partial_t u - \partial_x^2 u &= x + t & \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) &= e^x & (x \in \mathbb{R}). \end{cases}$$

b) $$\begin{cases} \partial_t u - 4\partial_x^2 u + u &= e^x & \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) &= x^2 & (x \in \mathbb{R}). \end{cases}$$

Solution: a) Instead of considering the whole problem, we split our equation into two sub-problems, namely

$$\begin{cases} \partial_t u_1 - \partial_x^2 u_1 &= 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ u_1(0, x) &= e^x & (x \in \mathbb{R}). \end{cases}$$

and

$$\begin{cases} \partial_t u_2 - \partial_x^2 u_2 &= x+t \quad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ u_2(0,x) &= 0 \qquad (x \in \mathbb{R}). \end{cases}$$

The solution of the first sub-problem by our formula is

$$u_1(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} \cdot e^{x-2\sqrt{t}\eta}\, d\eta = e^{x+t} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(\eta+\sqrt{t})^2}\, d\eta = e^{x+t},$$

in which we used that $\int_{-\infty}^{+\infty} e^{-\eta^2}\, d\eta = \sqrt{\pi}$ (see Exercise 3 on Practice 6).
We seek the solution of the second sub-problem using the Duhamel-principle. Our auxiliary problem is the following:

$$\begin{cases} \partial_t v - \Delta v &= 0 \qquad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ v(0,x) &= x+\tau \quad (x \in \mathbb{R}^n). \end{cases}$$

The solution of the above problem is

$$v(t,x;\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} (x+\tau-2\sqrt{t}\eta)\, d\eta =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} (x+\tau)\, d\eta - \frac{2\sqrt{t}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} \eta\, d\eta = x+\tau,$$

since the second integral is an integral of an odd function, so it is zero, and for the first we use Exercise 3 from Practice 6. Then the solution of the second sub-problem is

$$u_2(t,x) = \int_0^t (x+\tau)\, d\tau = tx + \frac{t^2}{2}.$$

By the linearity of the problem, the solution is the sum of the solutions of the two sub-problems so $u(t,x) = u_1(t,x) + u_2(t,x) = e^{x+t} + tx + \frac{t^2}{2}$.

b) First we transform our problem to a "regular" form. Let us substitute $2x$ instead of x, and let $z(t,x) = u(t,2x)$, then we get the following Cauchy-problem for z:

$$\begin{cases} \partial_t z - \partial_x^2 z + z &= e^{2x} \quad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ z(0,x) &= 4x^2 \quad (x \in \mathbb{R}). \end{cases}$$

Let us multiply the equation with e^t, and let $w(t,x) = e^t z(t,x)$, then we get the following Cauchy-problem for w (using that $w(0,x) = z(0,x)$):

$$\begin{cases} \partial_t w - \partial_x^2 w &= e^{2x+t} \quad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ w(0,x) &= 4x^2 \qquad (x \in \mathbb{R}). \end{cases}$$

This is now a problem we can solve. The two sub-problems in this case are

$$\begin{cases} \partial_t w_1 - \partial_x^2 w_1 &= 0 \qquad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ w_1(0,x) &= 4x^2 \quad (x \in \mathbb{R}), \end{cases}$$

and

$$\begin{cases} \partial_t w_2 - \partial_x^2 w_2 &= e^{2x+t} \quad \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ w_2(0,x) &= 0 \qquad (x \in \mathbb{R}). \end{cases}$$

The solution of the first sub-problem from the formula is

$$w_1(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} 4(x-2\sqrt{t}\eta)^2\, d\eta =$$

$$= 4x^2 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2}\, d\eta - 16\sqrt{t} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} \eta\, d\eta + 16t \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} \eta^2\, d\eta.$$

Observe that since the function $\eta \mapsto e^{-\eta^2}\eta$ is odd, the second integral on the right-hand side of the above equation is zero. The first integral equals $4x^2$, by using that

$\int_{-\infty}^{+\infty} e^{-\eta^2}\,d\eta = \sqrt{\pi}$ (see Exercise 3 on Practice 6). We can also compute the third integral:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\eta^2}\eta^2\,d\eta = \frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}e^{-\eta^2}\eta\right]_{-\infty}^{+\infty} + \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\eta^2}\,d\eta = \frac{1}{2},$$

in which we used that $e^{-\eta^2}\eta \to 0$ when $|\eta| \to +\infty$, since the decrease of the exponential is faster than the growth of the polynomial. In conclusion, $w_1(t,x) = 4x^2 + 8t$.

For the solution of the 2nd sub-problem, we define the following auxiliary problem:

$$\begin{cases} \partial_t v - \Delta v &= 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}, \\ v(0,x) &= e^{2x+\tau} & (x \in \mathbb{R}^n). \end{cases}$$

The solution of this one is

$$v(t,x;\tau) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\eta^2}\cdot e^{2x-4\sqrt{t}\eta+\tau}\,d\eta = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(\eta-2\sqrt{t})^2+4t+2x+\tau}\,d\eta =$$

$$= e^{2x+\tau+4t}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2}\,d\xi = e^{2x+\tau+4t}.$$

where we used the substitution $\xi = \eta - 2\sqrt{t}$. Then by the Duhamel-principle, the solution of the 2nd sub-problem is

$$u_2(t,x) = \int_0^t e^{2x+\tau+4(t-\tau)}\,d\tau = e^{2x+4t}\int_0^t e^{-3\tau}\,d\tau = -\frac{1}{3}e^{2x+4t}(e^{-3t}-1).$$

By linearity of the equation, the solution of the original problem is the sum of the solutions of the sub-problems, i.e. $w(t,x) = 4x^2 + 8t + \frac{1}{3}e^{2x+4t} - \frac{1}{3}e^{2x+t}$, and then $u(t,x) = x^2e^{-t} + 8te^{-t} + \frac{1}{3}e^{x+3t} - \frac{1}{3}e^{x}$.