Eighth practice

Hyperbolic equations

In this Practice we first prove the general form of solutions of a hyperbolic Cauchy-problem in the form

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) = f(t,x) & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,x) = g(x) & (x \in \mathbb{R}^n), \\ \partial_t u(0,x) = h(x) & (x \in \mathbb{R}^n). \end{cases} \tag{1}$$

For this, we split our general equation into three sub-problems. The first sub-problem is

$$\begin{cases} \partial_t^2 u_1(t,x) - \Delta u_1(t,x) = 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ u_1(0,x) = 0 & (x \in \mathbb{R}^n), \\ \partial_t u_1(0,x) = h(x) & (x \in \mathbb{R}^n). \end{cases}$$

The second sub-problem is

$$\begin{cases} \partial_t^2 u_2(t,x) - \Delta u_2(t,x) = f(t,x) & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ u_2(0,x) = 0 & (x \in \mathbb{R}^n), \\ \partial_t u_2(0,x) = 0 & (x \in \mathbb{R}^n). \end{cases}$$

The third sub-problem is

$$\begin{cases} \partial_t^2 u_3(t,x) - \Delta u_3(t,x) = 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ u_3(0,x) = g(x) & (x \in \mathbb{R}^n), \\ \partial_t u_3(0,x) = 0 & (x \in \mathbb{R}^n). \end{cases}$$

Then the solution of the general problem (1) is the sum of the solutions of the three sub-problems (since (1) is linear), so $u = u_1 + u_2 + u_3$.

1. Consider the following set of equations:

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ v(0,x) = 0 & (x \in \mathbb{R}^n), \\ \partial_t v(0,x) = f(\tau,x) & (x \in \mathbb{R}^n) \end{cases} \tag{2}$$

in which $f \in C(\mathbb{R}^+ \times \mathbb{R}^n)$ and $\tau \in \mathbb{R}_0^+$ is a parameter. Suppose that for all $\tau \in \mathbb{R}_0^+$, for the solution $v(\cdot,\cdot;\tau)$ of the above equation, $v, \partial_t^2 v, \Delta v \in C(\mathbb{R}_0^+ \times \mathbb{R}^n \times \mathbb{R}_0^+)$ holds. Then let us define function u in the following way:

$$u(t,x) = \int_0^t v(t-\tau, x; \tau)\, d\tau.$$

Prove that $\partial_t^2 u - \Delta u = f$ inside $\mathbb{R}^+ \times \mathbb{R}^n$, $u(0,x) = 0$ and $\partial_t u(0,x) = 0$ $(x \in \mathbb{R}^n)$, i.e. u is a *solution of the second sub-problem*! (Duhamel's principle)

Solution: Indeed, $u(0,x) = \int_0^0 (\dots) = 0$ $(x \in \mathbb{R}^n)$. Then using Exercise 6 from Practice 7, we get that

$$\partial_t u(t,x) = v(0,x;t) + \int_0^t \partial_t v(t-\tau, x; \tau)\, d\tau.$$

Since v is a solution of (2) with parameter $\tau = t$, then $v(0,x;t) = 0$, consequently $\partial_t u(0,x) = \int_0^0 (\dots) = 0$ $(x \in \mathbb{R}^n)$, so the initial conditions are fulfilled. Also,

$$\partial_t^2 u(t,x) = \partial_t v(0,x;t) + \int_0^t \partial_t^2 v(t-\tau, x; \tau)\, d\tau.$$

We know that $v(\cdot,\cdot;t)$ is the solution of (2) with parameter $\tau = t$, meaning that $\partial_t v(0,x;t) = f(t,x)$. Furthermore (because of the smoothness conditions, the derivation can be moved inside the integral),

$$\Delta u(t,x) = \int_0^t \Delta v(t-\tau, x; \tau)\, d\tau.$$

In conclusion,

$$\partial_t^2 u(t,x) - \Delta u(t,x) = f(t,x) + \int_0^t \left(\partial_t^2 v(t-\tau,x;\tau) - \Delta v(t-\tau,x;\tau)\right) d\tau. \tag{3}$$

The integrand on the right-hand side of (3) is zero, since $v(\cdot,\cdot;\tau)$ is a solution of (2), i.e. $\partial_t^2 v - \Delta v = 0$. So we proved that u is a solution of the second sub-problem.

2. Let w be the solution of the following problem:

$$\begin{cases} \partial_t^2 w - \Delta w = 0 & \text{inside } \mathbb{R}^+ \times \mathbb{R}^n, \\ w(0,x) = 0 & (x \in \mathbb{R}^n), \\ \partial_t w(0,x) = g(x) & (x \in \mathbb{R}^n). \end{cases}$$

Suppose that $w \in C^3(\mathbb{R}^+ \times \mathbb{R}^n) \cap C^2(\mathbb{R}_0^+ \times \mathbb{R}^n)$, and let us define $u(t,x) = \partial_t w(t,x)$ ($(t,x) \in {\mathbb{R}^+}_0 \times \mathbb{R}^n$). Prove that $\partial_t^2 u - \Delta u = 0$ inside $\mathbb{R}^+ \times \mathbb{R}^n$, $u(0,x) = g(x)$ and $\partial_t u(0,x) = 0$ ($x \in \mathbb{R}^n$), i.e. u is a *solution of the third sub-problem*!

Solution: Indeed, $u(0,x) = \partial_t w(0,x) = g(x)$ ($x \in \mathbb{R}^n$). Moreover, by the smoothness conditions, if $(t,x) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ then

$$\partial_t u(t,x) = \partial_t^2 w(t,x) = \Delta w(t,x),$$

consequently $\partial_t u(0,x) = \Delta w(0,x) = 0$ ($x \in \mathbb{R}^n$). Also, $\partial_t^2 u = \partial_t(\partial_t^2 w)$ and $\Delta u = \Delta \partial_t w = \partial_t(\Delta w)$, therefore $\partial_t^2 u - \Delta u = \partial_t(\partial_t^2 w - \Delta w) = 0$. In this way we proved that u is a solution of the 3rd sub-problem.

3. Solve the first sub-problem in the case $n = 1$.

Solution: From Exercise 1 (f) from Sheet 1, the solution is in the form $u(t,x) = F(x+t) + G(x-t)$ for some functions $F, G \in C^2(\mathbb{R})$. The form of functions F, G is determined by the initial and boundary conditions. From the initial conditions: $\partial_t u(t,x) = F'(x+t) - G'(x-t)$, so $h(x) = \partial_t u(0,x) = F'(x) - G'(x)$ ($x \in \mathbb{R}^n$). Furthermore, $0 = u(0,x) = F(x) + G(x)$ ($x \in \mathbb{R}^n$). Combining these two we get $\frac{1}{2}h(x) = F'(x)$, so $F(x) = \frac{1}{2}\int_0^x h(\xi)\, d\xi + c$ and $G(x) = -\frac{1}{2}\int_0^x h(\xi)\, d\xi - c$. In conclusion,

$$u(t,x) = \frac{1}{2}\int_0^{x+t} h(\xi)\, d\xi + c - \frac{1}{2}\int_0^{x-t} h(\xi)\, d\xi - c = \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\, d\xi.$$

4. Prove that in the case $n = 1$ and $g \in C^2(\mathbb{R})$, the solution of the third sub-problem is $u(t,x) = \frac{1}{2}(g(x+t) + g(x-t))$ ($(t,x) \in {\mathbb{R}^+}_0 \times \mathbb{R}$).

Solution: By the formulas of the solutions of the 2nd and 3rd sub-problems proved above,

$$u(t,x) = \partial_t \left(\frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi \right) = \frac{1}{2}\left(g(x+t) + g(x-t)\right).$$

Observe that we can now state the general form of the solutions of the hyperbolic Cauchy-problems in the case $n = 1$. By Exercise 3, the solution of the first sub-problem is

$$u_1 = \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\, d\xi.$$

By Exercise 4, the solution of the 3rd sub-problem is

$$u_3(t,x) = \frac{1}{2}(g(x+t) + g(x-t)).$$

Also, Exercise 1 states that the solution of the 2nd sub-problem can be computed from the solution of (2) by integration. The solution of equation (2) (by Exercise 3) is

$$v(t,x;\tau) = \frac{1}{2}\int_{x-t}^{x+t} f(\tau,\xi)\,d\xi,$$

meaning that by Exercise 1

$$u_2(t,x) = \int_0^t \frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau,\xi)\,d\xi\,d\tau.$$

In conclusion, the solution of the hyperblic Cauchy-problem is the sum of the solutions of the three sub-problems (because the equation is linear), i.e.

$$u(t,x) = \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau,\xi)\,d\xi\,d\tau + \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi. \tag{4}$$

This is the so-called *d'Alembert formula.*

5. Solve the following Cauchy-problem!

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = t - x & \text{inside } \mathbb{R}^+\times\mathbb{R}, \\ u(0,x) = \sin x & (x\in\mathbb{R}), \\ \partial_t u(0,x) = \cos x & (x\in\mathbb{R}). \end{cases}$$

Solution: The auxiliary problem corresponding to the second sub-problem is:

$$\begin{cases} \partial_t^2 v - \partial_x^2 v = 0 & \text{inside } \mathbb{R}^+\times\mathbb{R}, \\ v(0,x) = 0 & (x\in\mathbb{R}), \\ \partial_t v(0,x) = \tau - x & (x\in\mathbb{R}). \end{cases}$$

The solution of this one is

$$v(t,x;\tau) = \frac{1}{2}\int_{x-t}^{x+t} (\tau-\xi)\,d\xi = \frac{1}{2}\left[\tau\xi - \frac{\xi^2}{2}\right]_{\xi=x-t}^{x+t} = \tau t - tx.$$

Therefore the solution of the 2nd sub-problem is

$$u_2(t,x) = \int_0^t (t-\tau)(\tau-x)\,d\tau = \int_0^t (t\tau - tx - \tau^2 + \tau x)\,d\tau =$$

$$= \left[t\frac{\tau^2}{2} - \frac{\tau^3}{3} - tx\tau + \frac{\tau^2}{2}x\right]_{\tau=0}^{t} = \frac{t^3}{6} - \frac{t^2 x}{2}.$$

The auxiliary problems corresponding to the first and third sub-problems are:

$$\begin{cases} \partial_t^2 u_1 - \partial_x^2 u_1 = 0 & \text{inside } \mathbb{R}^+\times\mathbb{R}, \\ u_1(0,x) = \sin x & (x\in\mathbb{R}), \\ \partial_t u_1(0,x) = 0 & (x\in\mathbb{R}), \end{cases} \qquad \begin{cases} \partial_t^2 u_3 - \partial_x^2 u_3 = 0 & \text{inside } \mathbb{R}^+\times\mathbb{R}, \\ u_3(0,x) = 0 & (x\in\mathbb{R}), \\ \partial_t u_3(0,x) = \cos x & (x\in\mathbb{R}). \end{cases}$$

The solutions of these are $u_1(t,x) = \frac{1}{2}(\sin(x+t)+\sin(x-t))$ and $u_3(t,x) = \int_{x-t}^{x+t}\cos\xi\,d\xi = \frac{1}{2}(\sin(x+t)-\sin(x-t))$, respectively. In conclusion, the solution of the Cauchy-problem is $u(t,x) = \frac{t^3}{6} - \frac{t^2x}{2} + \sin(x+t)$.

6. Let u be the solution of the following problem:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{inside } \mathbb{R}^+\times\mathbb{R}^n, \\ u(0,x) = g(x) & (x\in\mathbb{R}), \\ \partial_t u(0,x) = h(x) & (x\in\mathbb{R}), \end{cases}$$

in which $g,\ h\in C(\mathbb{R})$. Show that if $\operatorname{supp} g, \operatorname{supp} h \subset [a,b]$, then $\operatorname{supp} u(t,\cdot) \subset [a-t, b+t]$ for all $t>0$. (So the wave propagates with a finite speed.)

Solution: By Exercises 3 and 4, we know that

$$u(t,x) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi.$$

It is clear that for $x \notin [a-t, b+t]$, the interval $[x-t, x+t]$ lies outside of interval $[a,b]$ (their end-points might be the same), meaning that if $\operatorname{supp} g \subset [a,b]$ and $\operatorname{supp} h \subset [a,b]$, then on the interval $[x-t, x+t]$ both g and h equals to zero. Then by the formula for the solution $u(t,x) = 0$, so $\operatorname{supp} u(t,\cdot) \subset [a-t, b+t]$. This result means that the effect of an initial wave concentrated on the interval $[a,b]$ after time t can only be seen on the interval $[a-t, b+t]$, so the speed of the wave is finite (here 1), contrary to the case of heat equation (in which it is infinite).

7. Prove for the case of $n = 1$, the solution of the hyperbolic Cauchy-problem depends continuously on h in the following sense: if h_1, $h_2 \in C(\mathbb{R})$, for which

$$|h_1(x) - h_2(x)| \le \varepsilon \ (x \in \mathbb{R}),$$

then for the corresponding solutions of the Cauchy-problem u_1, u_2 we get that

$$|u_1(t,x) - u_2(t,x)| \le \varepsilon t \ \ ((t,x) \in {\mathbb{R}^+}_0 \times \mathbb{R}).$$

Solution: It is clear that

$$|u_1(t,x) - u_2(t,x)| \le \frac{1}{2}\int_{x-t}^{x+t} |h_1(\xi) - h_2(\xi)|\,d\xi \le \frac{1}{2}\int_{x-t}^{x+t} \varepsilon\,d\xi = \varepsilon t.$$

8. Show that in the case of $n = 1$, the solution u of the hyperbolic Cauchy-problem depends continuously on f in the following sense: if $f_1, f_2 \in C(\mathbb{R}^+ \times \mathbb{R})$, for which

$$|f_1(t,x) - f_2(t,x)| \le \varepsilon \ \ ((t,x) \in \mathbb{R}^+ \times \mathbb{R}),$$

then

$$|u_1(t,x) - u_2(t,x)| \le \frac{\varepsilon t^2}{2} \ \ ((t,x) \in \mathbb{R}_0^+ \times \mathbb{R}).$$

Solution: From the D'Alembert formula (4) it is easy to see that

$$|u_1(t,x) - u_2(t,x)| \le \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} |f_1(\tau,\xi) - f_2(\tau,\xi)|\,d\xi\,d\tau \le \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \varepsilon\,d\xi\,d\tau =$$

$$= \int_0^t \varepsilon(t-\tau)\,d\tau = \varepsilon\left[-\frac{(t-\tau)^2}{2}\right]_0^t = \frac{\varepsilon t^2}{2}.$$

9.* Let $u \in C^2(\mathbb{R}_0^+ \times [0{,}1])$ be such a solution of the one-dimensional wave equation $\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0$ for which $u(t{,}0) = u(t{,}1) = 0$ for every $t > 0$ (this is a rod with fixed ends). Show that then in this case the following function (mechanical energy) does not depend on t:

$$E(t) = \frac{1}{2}\int_0^1 \left[(\partial_t u(t,x))^2 + (\partial_x u(t,x))^2\right]dx.$$

Solution: The solution can be submitted.