Nineth practice

Elliptic equations

1.* Show that the Laplace equation $\Delta u = 0$ is rotation-invariant in $\mathbb{R}^n$, meaning that if we have an $n \times n$ orthogonal matrix Q, then for the function $v(x) = u(Qx)$ $(x \in \mathbb{R}^n)$ the equation $\Delta v = 0$ also holds.

Solution: The solution can be submitted.

2. We seek the solutions of the equation $\Delta u = 0$ in the form $u(x) = v(|x|)$ $(x \in \mathbb{R}^n)$ (the solution is radially symmetric), in which $v \colon \mathbb{R}_0^+ \to \mathbb{R}$.

Solution: For the sake of simplicity let $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$, and we seek the solutions of equation $\Delta u = 0$ in the form $u(x) = v(r)$, in which we would like to acquire the function $v \colon \mathbb{R}^+{}_0 \to \mathbb{R}$. By the differentiation rule of the composite functions:

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r} \quad (x \neq 0),$$

then in the case of $i = 1, \ldots, n$:

$$\partial_i u(x) = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r},$$

(in which „$'$” is the differentiation by r), and then

$$\partial_i^2 u(x) = v''(r)\frac{\partial r}{\partial x_i}\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}\frac{\partial r}{\partial x_i}\right) = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

Consequently,

$$\Delta u(x) = \sum_{i=1}^n \partial_i^2 u(x) = \sum_{i=1}^n \left[v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right] =$$

$$= v''(r)\frac{\sum_{i=1}^n x_i^2}{r^2} + \frac{1}{r}v'(r)\left(n - \frac{\sum_{i=1}^n x_i^2}{r^2}\right) = v''(r) + \frac{n-1}{r}v'(r).$$

This means that equation $\Delta u = 0$ holds if and only if

$$v''(r) + \frac{n-1}{r}v'(r) = 0.$$

This is a separable ordinary differential equation in v', so we can get its solutions easily. By moving its terms we get

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r}.$$

By integrating both sides, we get $\log|v'(r)| = (1-n)\log r + c$, meaning that $v'(r) = Cr^{1-n}$, in which C is an arbitrary constant (we can get back the constant zero solution which we lost when we divided the equation). Therefore, in the case of $r > 0$:

$$v(r) = \begin{cases} a\log r + b, & \text{if } n = 2, \\ \frac{a}{r^{n-2}} + b, & \text{if } n \geq 3, \end{cases}$$

in which a, b are constants. Note that if $n = 2$, then with the choices $a = \frac{1}{2\pi}$, $b = 0$, and in the case of $n \geq 3$ with the choices $a = \frac{1}{n(n-2)\alpha(n)}$, $b = 0$ (in which $\alpha(n)$ is the volume of the n-dimensional sphere) we get the fundamental solutions of the Laplace-equation. Note that we have also proved that in the case of radially symmetric functions the operator Δ can be written in the form $v''(r) + \frac{n-1}{r}v'(r)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and $p \in C^1(\overline{\Omega})$, for which $p(x) \geq m > 0$ for all $x \in \overline{\Omega}$. Let us define the second order differential operator $Lu := \operatorname{div}(p\nabla u) = \sum_{i=1}^{n} \partial_i(p\, \partial_i u)$. Prove that L is a uniformly elliptic operator!

   Solution: From the smoothness conditions:

$$Lu = \sum_{i=1}^{n} \partial_i(p\, \partial_i u) = \sum_{i=1}^{n} p\partial_i^2 u + \sum_{i=1}^{n} \partial_i p\, \partial_i u.$$

   The matrix $A(x)$ in the main term of operator L is diagonal, the elements in its main diagonal are $p(x) \geq -m > 0$, so the eigenvalues are negative, meaning that it is elliptic in all the points of Ω. Also, since $p \in C^1(\overline{\Omega})$, there is such a constant $M > 0$, for which $m \leq p(x) \leq M$ for all $x \in \overline{\Omega}$, meaning that

$$m|\xi|^2 \leq p(x)|\xi|^2 = \langle A(x)\xi, \xi\rangle = p(x)|\xi|^2 \leq M|\xi|^2$$

   for all $\xi \in \mathbb{R}^n$, so the operator is uniformly elliptic on Ω. Note that the reason for the negative sign is described in the next Exercise.

4. Let operator L be as defined in Exercise 3.

   a) Let $D(L) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega})\colon u|_{\partial\Omega} = 0,\ Lu \in L^2(\Omega)\}$. Prove that in this case the operator $L\colon L^2(\Omega) \hookrightarrow L^2(\Omega)$ is symmetric, i.e. $\langle Lu, v\rangle_{L^2(\Omega)} = \langle u, Lv\rangle_{L^2(\Omega)}$ for all $u, v \in D(L)$, and L is strictly positive, i.e. $\langle Lu, u\rangle_{L^2(\Omega)} > 0$ for all $u \in D(L)$, $u \not\equiv 0$.

   b) Let $D(L) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega})\colon \partial_\nu u|_{\partial\Omega} = 0,\ Lu \in L^2(\Omega)\}$. Prove that in this case the operator $L\colon L^2(\Omega) \hookrightarrow L^2(\Omega)$ is symmetric, i.e. $\langle Lu, v\rangle_{L^2(\Omega)} = \langle u, Lv\rangle_{L^2(\Omega)}$ for all $u, v \in D(L)$, and L is positive, i.e. $\langle Lu, u\rangle_{L^2(\Omega)} \geq 0$ for all $u \in D(L)$, and equality can only hold if $u \equiv c \in \mathbb{R}$.

   Solution:

   a) We use the second Green theorem, which stated that for a "sufficiently nice" domain and functions $u, v \in D(L)$ (note that in this case $Lu = -\operatorname{div}(p \operatorname{grad} u)$):

$$\int_\Omega (vLu - uLv) = -\int_{\partial\Omega} p(v\partial_\nu u - u\partial_\nu v)\, d\sigma.$$

   Since $u, v \in D(L)$, then $u|_{\partial\Omega} = 0$ and $v|_{\partial\Omega} = 0$, so the right-hand side is zero. Therefore, by the homogeneous boundary conditions $\langle Lu, v\rangle - \langle u, Lv\rangle = 0$ for all $u, v \in D(L)$. Also, by the choice $v = u$, in the case of $u \in D(L)$ by using the first Green theorem we get

$$\langle Lu, u\rangle_{L^2(\Omega)} = \int_\Omega uLu = \int_\Omega p\,|\operatorname{grad} u|^2 - \int_{\partial\Omega} pv\partial_\nu u \geq m\int_\Omega |\operatorname{grad} u|^2 \geq 0.$$

   Equality holds on the right-hand side of this inequality if and only if $\operatorname{grad} u = 0$, so $u \equiv c \in \mathbb{R}$. By the homogeneous boundary condition inside $D(L)$, we get $c = 0$, so $u \equiv 0$. In this way we proved that L is a strictly positive operator on $D(L)$.

   b) We prove this one similarly as part a): by using the second Green theorem and the boundary condition we get that

$$\int_\Omega vLu - \int_\Omega uLv = 0.$$

   Also, in the case of $u = v$, $u \not\equiv c \in \mathbb{R}$, by the first Green theorem

$$\langle Lu, u\rangle_{L^2(\Omega)} = \int_\Omega uLu = \int_\Omega p|\operatorname{grad} u|^2 \geq m\int_\Omega |\operatorname{grad} u|^2 \geq 0.$$

   Equality holds on the right-hand side of this inequality if and only if $\operatorname{grad} u = 0$, so $u \equiv c \in \mathbb{R}$ (and the constant functions are all in $D(L)$).

5. Show that the Dirichlet problem can have at most one solution inside $C^2(\Omega)\cap C^1(\overline{\Omega})$, and the solutions of the Neumann-problem can only differ from each other in a constant.

   Solution: If we have two solutions in $C^2(\Omega)\cap C^1(\overline{\Omega})$, then its difference, which is also in $C^2(\Omega)\cap C^1(\overline{\Omega})$, is also a solution of the homogeneous problem, so $L(u_1-u_2)=0$ and $u_1-u_2|_{\partial\Omega}$, and consequently $\langle L(u_1-u_2), u_1-u_2\rangle = 0$. By Exercise 4 we know that this can only hold in the case of the Dirichlet condition if $u_1-u_2=0$, or in the case of the Neumann boundary if u_1-u_2 is constant. In other words, the solution of the Dirichlet-problem is unique in $C^2(\Omega)\cap C^1(\overline{\Omega})$, and the solutions of the Neumann-problem in this function space can only differ from each other in a constant, i.e. if u_1 and u_2 are two different solutions, then $u_1=u_2+\text{const}$. Note that the solution of the Dirichlet problem is also unique in $u\in C^2(\Omega)\cap C(\overline{\Omega})$, which can be proved using the maximum principle. The existence of a solution is a much harder question (it will be discussed on the last lecture of this semester).

6. Let operator L be the same as in Exercise 3, and $D(L)=\{u\in C^2(\Omega)\cap C^1(\overline{\Omega})\colon \partial_\nu u|_{\partial\Omega}=0,\ Lu\in L^2(\Omega)\}$. Prove that if

$$\begin{cases} Lu &= f \quad \text{in } \Omega,\\ \partial_\nu u|_{\partial\Omega} &= 0, \end{cases}$$

   then $\displaystyle\int_\Omega f = 0$.

   Solution: By applying the first Green theorem for the functions $u\in D(L)$, $v\in C^1(\overline{\Omega})$, and then using the homogeneous boundary condition, we get that

$$\int_\Omega vf = \int_\Omega vLu = \int_\Omega p\langle \operatorname{grad} u, \operatorname{grad} v\rangle.$$

   Choose v to be the constant 1 function, this is in $C^1(\overline{\Omega})$, then by the above equality $\displaystyle\int_\Omega f = 0$.

   Note that this proof lies on the symmetric property of L. Indeed, for an arbitrary symmetric operator $L\colon D(L)\to H$, $\operatorname{Ran}(L)\subset \operatorname{Ker}(L)^\perp$ holds, since for all $u\in D(L)$ and $v\in\operatorname{Ker}(L)$, $\langle Lu,v\rangle = \langle u, Lv\rangle = 0$. Note that the above condition is also sufficient for the existence of solutions, see the Fredholm alternative theory (possibly mentioned in the Lecture, if we have enough time).

7. Let $\Omega=(0,1)^2\subset\mathbb{R}^2$. Prove that the problem

$$\begin{cases} \Delta u = 1 \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0, \end{cases}$$

   has no $u\in C^2(\overline{\Omega})$ solution.

   Solution: Consider one of the angles of the domain, e.g. the origin. On both of the x and y axis we have $u=0$ because of the boundary condition. Then, since u is twice differentiable on the closure of the domain, $\partial_x u(x,0)=\partial_y u(0,y)=0$, so $\partial_x^2 u(0,0) = = \partial_y^2 u(0,0) = 0$, meaning that $\Delta u(0,0) = \partial_x^2 u(0,0)+\partial_y^2 u(0,0) = 0$. We get that if the boundary condition holds for u (and is twice differentiable on the closure of the domain), then $\Delta u(0,0)=0$. So for u, the equation $\Delta u = 1$ does not hold in all of the interior of the domain, because in this case by continuity this should also hold at the origin, which is a contradiction.

8. Let $B_1(0)$ be an open sphere centered at the origin with radius 1. For which value of $\alpha\in\mathbb{R}$ does the following boundary-value problem have a solution $u\in C^2(B_1(0))\cap C^1(\overline{B_1(0)})$? Give the solutions!

$$\begin{cases} \Delta u &= \alpha \quad B_1(0)\text{-ban},\\ \partial_\nu u|_{\partial B_1(0)} &= 1. \end{cases}$$

   Solution: Apply the first Green formula to functions u and constant 1. Then by $\operatorname{grad} 1 = 0$, we get

$$\int_{B_1(0)} 1\cdot\Delta u = \int_{\partial B_1(0)} 1\cdot \partial_\nu u\, d\sigma.$$

By substituting $\Delta u = \alpha$ and $\partial_\nu|_{\partial B_1(0)} u = 1$ into the above formula, we get

$$\int_{B_1(0)} \alpha = \int_{\partial B_1(0)} 1 \, d\sigma.$$

On the left-hand side of the equation we have the integral of the constant α function on the unit disc, which is actually the area of the unit disc times α, i.e. $\alpha\pi$. On the right-hand side we have the integral of the constant one function taken on the circumference of the unit disc, which is the length of the circumference, i.e. 2π. These two are equal, meaning that $\alpha\pi = 2\pi$, so $\alpha = 2$. Consequently, the problem can have a solution in $C^2(B_1(0)) \cap C^1(\overline{B_1(0)})$ only if $\alpha = 2$.

Observe that such a solution actually exists, since for function

$$u(x,y) = \frac{1}{2}(x^2+y^2) = \frac{1}{2}r^2$$

we have $\Delta u = 2$ and $\partial_\nu u|_{\partial B_1(0)} u = \partial_r u|_{r=1} = 1$. By Exercise 5 we know that the solutions in $C^2(B_1(0)) \cap C^1(\overline{B_1(0)})$ of the Neumann problem differ in a constant from each other, so if $\alpha = 2$, then all of the solutions are in the form $u(x,y) = \frac{1}{2}(x^2+y^2)+c$, in which $c \in \mathbb{R}$ is arbitrary.

We could have also solved this problem in the following way. Consider the function $v(x,y) := u(x,y) - \frac{1}{2}r^2$. Then for v the homogeneous Neumann condition holds, and also $\Delta v = \alpha - 2$. From Exercise 6 we know that in this case $\int_{B_1(0)}(\alpha-2) = 0$ holds, so $\alpha - 2 = 0$, but then $v = c$, since the solutions of the homogeneous Neumann problem are the constant functions.

9. Let $B_1(0)$ be the open unit disc centered at the origin, and $T := \{(x,y) \in \mathbb{R}^2 : x^2 + x + 2y^2 < 1\}$ (the interior of an ellipse). Solve the following boundary-value problems!

   a) $\begin{cases} \Delta u = x + y & \text{in } B_1(0), \\ u|_{\partial B_1(0)} = 0. \end{cases}$

   b) $\begin{cases} \Delta u = x & \text{in } B_1(0), \\ u|_{\partial B_1(0)} = y^2. \end{cases}$

   c) $\begin{cases} \Delta u = 1 & \text{in the domain } T, \\ u|_{\partial T} = x^2. \end{cases}$

Solution:

The main idea of these exercises is to first search the solution in a form that suffices the boundary condition, and it is also a polynomial which has an order $n+2$ where n is the order of the right-hand side of the equation (it is a polynomial in all there types: you can also expect a polynomial in the midterm).

a) Seek solution u in the form $u(x,y) = (x^2+y^2-1)(ax+by+c)$, since then the boundary condition is satisfied. Then $\Delta u(x,y) = 8ax + 8by + 4c$. By the conditions, $a = b = \frac{1}{8}$ and $c = 0$. Consequently, the solution of the problem is $u(x,y) = \frac{x+y}{8}(x^2+y^2-1)$.

b) Seek solution u in the form $u(x,y) = (x^2+y^2-1)(ax+by+c) + y^2$ since then the boundary condition is satisfied. Then $\Delta u(x,y) = 8ax + 8by + 4c + 2$, so $a = \frac{1}{8}$, $b = 0$ and $c = -\frac{1}{2}$. Therefore, $u(x,y) = (x^2+y^2-1)(\frac{1}{8}x - \frac{1}{2}) + y^2$.

c) Seek solution u in the form $u(x,y) = (x^2+x+2y^2-1)c + x^2$ since then the boundary condition is satisfied. Then $\Delta u(x,y) = 6c+2$, so $c = -\frac{1}{6}$. Therefore, $u(x,y) = -\frac{1}{6}(x^2 + x + 2y^2 - 1) + x^2$.