First midterm 2023

Partial differential equations

Solutions

1. **Find the solution $u \in C^2(\mathbb{R}^2)$ of the following Cauchy problem!**

$$\begin{cases} \partial_{xy} u(y,x) = x + \sin(y), \\ u(0,y) = e^y, \\ \partial_x u(x{,}0) = -1. \end{cases}$$

Solution: By integrating the equation:

$$\partial_{xy} u(y,x) = x + \sin(y)$$

$$\partial_x u(x,y) = xy - \cos(y) + f(x)$$

(2 points) Now by using the second boundary condition:

$$\partial_x u(x{,}0) = -\cos(0) + f(x) = -1$$

meaning that $f(x) \equiv 0$. (2 points) Then, from

$$\partial_x u(x,y) = xy - \cos(y)$$

we get

$$u(x,y) = \frac{x^2 y}{2} - x\cos(y) + g(y)$$

(2 points) From the first boundary condition:

$$u(0,y) = g(y) = e^y,$$

(2 points) meaning that the final solution is

$$u(x,y) = \frac{x^2 y}{2} - x\cos(y) + e^y.$$

(2 points)

2. **Give all such $u \in C^2(\mathbb{R}^2)$ functions for which both $\Delta u = 0$ and $\Delta(\cosh(u)) = 0$ hold. (Note that you do not need to use complex analysis.)**

Solution: Let us rewrite the term on the left-hand side of the second equation! Then

$$\Delta(\cosh(u)) = 0 = \partial_{xx}(\cosh(u)) + \partial_{yy}(\cosh(u)) = \partial_x(\sinh(u)\partial_x u) + \partial_y(\sinh(u)\partial_y u) =$$

$$= \cosh(u)(\partial_x u)^2 + \sinh(u)\partial_x^2 u + \cosh(u)(\partial_y u)^2 + \sinh(u)\partial_y^2 u =$$

$$= \cosh(u)\left[(\partial_x u)^2 + (\partial_y u)^2\right] + \sinh(u)\Delta(u) =$$

(6 points) Now we use the fact that $\Delta u = 0$:

$$= \cosh(u)\left[(\partial_x u)^2 + (\partial_y u)^2\right] = 0$$

(2 points) Since $\cosh(u) > 0$, $(\partial_x u)^2 + (\partial_y u)^2 = 0$. So $\partial_x u = 0$ and $\partial_y u = 0$, which can only hold if $u = c$, $c \in \mathbb{R}$, so the solution is the set of constant functions. (2 points)

3. **Give all the solutions of the following first order partial differential equation which equal $2x$ on the x-axis:**

$$\partial_x u(x,y) + e^y \partial_y u(x,y) = 2$$

Solution: This is a quasi-linear equation, so the auxiliary equation has the form

$$\partial_x v(x,y,u) + e^y v(x,y,u) + 2\partial_u v(x,y,u) = 0,$$

and the corresponding characteristic equation is

$$\begin{cases} x'(t) = 1, \\ y'(t) = e^y, \\ \hat{u}'(t) = 2. \end{cases}$$

(2 points) It can be seen that $x(t) + e^{-y(t)}$ is a first integral, since

$$x'(t)e^{y(t)} = y'(t)$$
$$x'(t) = y'(t)e^{-y(t)}$$
$$\left[x(t) + e^{-y(t)}\right]' = 0$$

(2 points) Also, $u(t) + 2e^{-y(t)}$ is also a first integral, since

$$u'(t)e^{y(t)} = 2y'(t)$$
$$u'(t) = 2y'(t)e^{-y(t)}$$
$$\left[u(t) + 2e^{-y(t)}\right]' = 0$$

(2 points) This means that our solution is in the form

$$u(t) + 2e^{-y(t)} = \Psi\left(x(t) + e^{-y(t)}\right),$$
$$u = \Psi\left(x(t) + e^{-y(t)}\right) - 2e^{-y(t)}.$$

(2 points) The initial condition means that $u(x,0) = 2x$, so by using this one:

$$u(x,0) = \Psi\left(x(t) + 1\right) - 2 = 2x,$$

meaning that $\Psi(z) = 2z$, so the final solution is

$$u(x,y) = 2\left(x + e^{-y}\right) - 2e^{-y}.$$

(2 points)

4. **Give such a two-variable non-constant polynomial $a(x,y)$ for which the second-order differential operator**

$$Lu = \partial_x^2 u + \partial_{xy} u + a(x,y)\partial_y^2 u$$

is hyperbolic below the curve $y = -4x^4$, it is elliptic between the curves $y = -4x^4$ and $y = 4x^4$ and is hyperbolic above the curve $y = 4x^4$. (Hint: search for a polynomial in the form $a(x,y) = ax^8 + by^2 + c$.)

Solutions: The matrix associated with this operator is

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & a(x,y) \end{pmatrix}$$

(2 points) Then the determinant of this matrix is

$$\det(A) = a(x,y) - \frac{1}{4}.$$

(2 points) Our goal is that this expression is positive if $-4x^4 < y < 4x^4$, and is negative for $y < -4x^4$ and $y > 4x^4$. (2 points)

Let us choose $a(x,y)$ in a way that (for a fixed x value) it has roots at $y = -4x^4$ and $y = 4x^4$, and it is negative for $y = 0$ (when $x \neq 0$). If we choose

$$a(x,y) = \left(16x^8 - y^2 + \frac{1}{4}\right),$$

then

$$\det(A) = F(x,y) = \left(16x^8 - y^2 + \frac{1}{4}\right) - \frac{1}{4} = 16x^8 - y^2,$$

for which

$$F(x, 4x^4) = 0,$$
$$F(x, -4x^4) = 0.$$

Moreover, if $y > 4x^4$ then $F(x,y) = 16x^8 - y^2 < 0$, and it is also negative when $y < 4x^4$, and it is positive between the two curves. (4 points)

5. **For an arbitrary, fixed $\phi \in \mathcal{D}(\mathbb{R})$ function let us define**

$$\phi_j(x) := \frac{1}{j!}\phi(2023j - jx) \qquad (x \in \mathbb{R}, j = 1, 2, \dots).$$

Is this sequence convergent in the $\mathcal{D}(\mathbb{R})$ set? If yes, prove it, if not, give a counterexample!

Solution: Let us first consider the supports of these functions. It is clear that the term $\frac{1}{j!}$ does not affect the support, and if $x \in \text{supp}(\phi)$, then $(2023 - x/j) \in \text{supp}(\phi_j)$, meaning that the support of the functions gets smaller and smaller as $j \to \infty$ (the 2023 term only shifts it), so there is a compact set which contains all of the supports. (4 points)

Since we have the term $\frac{1}{j!}$, the functions ϕ_j tend to zero uniformly as $j \to \infty$. (3 points)

Let us consider the nth derivative of the sequence:

$$\phi_j^{(n)}(x) = (-1)^n \, \frac{j^n}{j!} \, \phi^{(n)}(2023j - jx)$$

It is a well known fact from analysis that the sequence $\frac{j^n}{j!}$ tends to zero as $j \to \infty$, meaning that the sequence is convergent. (3 points)

6. **Let $H \subset \mathbb{R}^2$ be the triangle on the plane with its vertices located at (0,0), (0,1) and (1,1). Let us define the distribution $u : \mathcal{D}(H) \to \mathbb{R}$ in the following way:**

$$u(\phi) := \int_0^1 \int_0^x e^y \phi(x, y) dy dx \qquad (\phi \in \mathcal{D}(H))$$

Show that

$$\partial_x u + \partial_y u = u.$$

(You don't have to show that it is a distribution.)

Solution: Let us calculate the two derivatives!

$$\partial_x u = -\int_0^1 \int_0^x e^y \, \partial_x \phi(x, y) dy dx = -\int_0^1 \int_y^1 e^y \, \partial_x \phi(x, y) dx dy =$$

$$= -\int_0^1 e^y \, [\phi(x, y)]_{x=y}^1 dy = -\int_0^1 e^y \, \phi(1, y) - e^y \, \phi(y, y) dy = \int_0^1 e^y \, \phi(y, y) dy$$

(4 points)

$$\partial_y u = -\int_0^1 \int_0^x e^y \, \partial_y \phi(x, y) dy dx =$$

By integration by parts:

$$= -\int_0^1 \left([e^y \, \phi(x, y)]_{y=0}^x - \int_0^x e^y \, \phi(x, y) \right) dx = -\int_0^1 \left([e^x \, \phi(x, x) - \phi(x,0)] - \int_0^x e^y \, \phi(x, y) \right) dx =$$

$$= -\int_0^1 e^x \, \phi(x, x) dx + \int_0^1 \int_0^x e^y \phi(x, y) dy dx$$

(5 points) By adding these things up:

$$\partial_x u + \partial_y u = \int_0^1 e^y \, \phi(y, y) dy - \int_0^1 e^x \, \phi(x, x) dx + \int_0^1 \int_0^x e^y \phi(x, y) dy dx =$$

$$= \int_0^1 \int_0^x e^y \phi(x, y) dy dx = u.$$

(1 point)

7. **Let $f \in C(\mathbb{R})$ and assume that its distributional derivative is a regular distribution, e.g. there is such a function $g \in L^1_{loc}(\mathbb{R})$ for which $(T_f)' = T_g$. Show such a function f that its distributional derivative has infinitely many jumps.**

Solution: Let us consider the function

$$f(x) = \begin{cases} \{x\}, & \text{if } 2k < x < 2k + 1, \ \forall k \in \mathbb{Z}, \\ 1 - \{x\}, & \text{if } 2k + 1 < x < 2k + 2, \ \forall k \in \mathbb{Z}. \end{cases}$$

Then its derivative (according to the theorem) is

$$g(x) = \begin{cases} 1, & \text{if } 2k < x < 2k + 1, \ \forall k \in \mathbb{Z}, \\ -1, & \text{if } 2k + 1 < x < 2k + 2, \ \forall k \in \mathbb{Z}, \end{cases}$$

which has infinitely many jumps. (10 points)