Second midterm

Partial differential equations, 2023

Solutions

1. **Solve the following parabolic Cauchy-problem!**

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) = x^3 + \frac{1}{2}t^2, & \text{on } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = 0, & (x \in \mathbb{R}). \end{cases}$$

Solution: Let us introduce the auxiliary problem

$$\begin{cases} \partial_t \tilde{w}(t,x) - \partial_x^2 \tilde{w}(t,x) = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}, \\ \tilde{w}(0,x) = x^3 + \frac{1}{2}\tau^2, & (x \in \mathbb{R}). \end{cases}$$

The solution of this Cauchy-problem is

$$\tilde{w}(t,x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} \left((x - 2\sqrt{t}\eta)^3 + \frac{1}{2}\tau^2 \right) d\eta =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} \left(x^3 - 6x^2\sqrt{t}\eta + 12xt\eta^2 - 8t\sqrt{t}\eta^3 \right) d\eta + \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} \frac{1}{2}\tau^2 d\eta =$$

$$= \frac{1}{\sqrt{\pi}} \left[x^3 \int_{\mathbb{R}} e^{-\eta^2} d\eta - 6x^2\sqrt{t} \int_{\mathbb{R}} e^{-\eta^2} \eta d\eta + 12xt \int_{\mathbb{R}} e^{-\eta^2} \eta^2 d\eta - 8t\sqrt{t} \int_{\mathbb{R}} e^{-\eta^2} \eta^3 d\eta + \right] + \frac{1}{2}\tau^2 =$$

The second and the fourth functions are odd, so the integrals are zero. Also, $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} d\eta = 1$ and $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \eta^2 d\eta = \frac{1}{2}$, meaning that

$$\tilde{w}(t,x) = x^3 + 6xt + \frac{1}{2}\tau^2.$$

Then by the Duhamel principle:

$$u(t,x) = \int_0^t \tilde{w}(t-\tau, x) d\tau = \int_0^t x^3 + 6x(t-\tau) + \frac{1}{2}\tau^2 d\tau = tx^3 + 6xt^2 + \int_0^t -6x\tau + \frac{1}{2}\tau^2 d\tau =$$

$$= tx^3 + 6xt^2 + 6x \left[-\frac{\tau^2}{2} \right]_{\tau=0}^t + \left[\frac{\tau^3}{6} \right]_{\tau=0}^t = tx^3 + 6xt^2 - 3xt^2 + \frac{t^3}{6}$$

So the solution is

$$u(x,t) = tx^3 + 3xt^2 + \frac{t^3}{6}.$$

2. **Let $g \in C^1(\mathbb{R})$ be an even function. Is it true that for the solution u of the hyperbolic equation**

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, & \textbf{on } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = g(x), & (x \in \mathbb{R}), \\ \partial_t u(0,x) = 0, & (x \in \mathbb{R}), \end{cases}$$

the function $x \to u(t,x)$ is also an even function for any fixed $t > 0$?

Solution: According to the well-known formula,

$$u(t,x) = \frac{1}{2}(g(x+t) + g(x-t)).$$

Function g is even, meaning that $g(x) = g(-x)$. Then,

$$u(t,-x) = \frac{1}{2}(g(-x+t) + g(-x-t)) =$$

Now we use that g is even:

$$= \frac{1}{2}(g(x-t) + g(x+t)) = u(t,x)$$

so the solution u is also even in x.

3. **Let $[a,b] \subset \mathbb{R}$ be a bounded interval and $p \in C^1([a,b])$, for which $p(x) \geq m > 0$ for every $x \in [a,b]$. Let us define the following operator $L : L^2(a,b) \hookrightarrow L^2(a,b)$ in the following way:**

$$D(L) := \{u \in C^2(a,b) \cap C^1([a,b]) : u(a) = 0, u'(b) + u(b) = 0, Lu \in L^2(\Omega)\}, \qquad Lu := -(p\,u')'.$$

Show that then this $L : L^2(\Omega) \hookrightarrow L^2(\Omega)$ operator is strictly positive, meaning that $\langle Lu, u\rangle_{L^2(\Omega)} > 0$ for every $u \in D(L)$, $u \neq 0$.

Solution:

$$\langle Lu, u\rangle_{L^2(\Omega)} = -\int_a^b (p\,u')'u =$$

Using partial integration:

$$= -\left(p(b)u'(b)u(b) - p(a)u'(a)u(a) - \int_a^b pu'u'\right) =$$

By the boundary conditions we have $u(a) = 0$, and also $u'(b) = -u(b)$:

$$= p(b)(u(b))^2 + \int_a^b p(u')^2 \geq m\int_a^b (u')^2$$

this is positive if $u' \neq 0$, which can only hold if u is a constant function, but then by the boundary condition we have $u = 0$, which is a contradiction, so we got the statement.

4. **Let $\Omega \subset \mathbb{R}^n$, $\Omega = B(0{,}4) \setminus \overline{B(0{,}2)}$ (where $B(0,r)$ is the ball centered at zero with radius r). Then for which $\alpha \in \mathbb{R}$ does the following problem have a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$?**

$$\begin{cases} \Delta u & = 1, \qquad (x \in \Omega) \\ \partial_\nu u|_{\partial\Omega} & = \alpha. \end{cases}$$

Solution: Use the first Green formula with $v \equiv 1$:

$$\int_\Omega 1 \cdot \Delta u = \int_{\partial\Omega} \partial_\mu u \; d\sigma$$
$$\int_\Omega 1 = \int_{\partial\Omega} \alpha \; d\sigma$$
$$16\pi - 4\pi = \alpha(8\pi + 4\pi)$$
$$\alpha = 1$$

So the problem has only solutions for $\alpha = 1$.

5. **Let Ω be the ball centered at zero with radius 1, and solve the following elliptic boundary-value problem!**

$$\begin{cases} \Delta u & = x + 2y, \qquad (x \in \Omega) \\ u|_{\partial\Omega} & = 3y. \end{cases}$$

Solution: Let us seek the solution in the form

$$u(x,y) = (x^2 + y^2 - 1)(ax + by + c) + 3y =$$

$$= ax^3 + axy^2 - ax + bx^2y + by^3 - by + cx^2 + cy^2 - c + 3y$$

Then

$$\Delta(u(x,y)) = 6ax + 2ax + 2by + 6by + 2c + 2c = 8ax + 8by + 4c$$

For this to be equal to the right-hand side, we need $a = \frac{1}{8}$, $b = \frac{1}{4}$ and $c = 0$, so the solution is in the form

$$u(x,y) = (x^2 + y^2 - 1)\left(\frac{x}{8} + \frac{y}{8}\right) + 3y$$

6. **Solve the following parabolic (mixed) problem!**

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) = \sin(t)\sin(3x) & ((t,x) \in \mathbb{R}^+ \times (0,\pi)) \\ u(0,x) = \sin(4x), & (x \in [0,\pi]) \\ u(t,0) = u(t,\pi) = 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

Solution: Let us split our problem into two, easier sub-problems:

$$\begin{cases} \partial_t u_1(t,x) - \partial_x^2 u_1(t,x) = \sin(t)\sin(3x) & ((t,x) \in \mathbb{R}^+ \times (0,\pi)) \\ u_1(0,x) = 0, & (x \in [0,|\pi]) \\ u_1(t,0) = u_1(t,\pi) = 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

and

$$\begin{cases} \partial_t u_2(t,x) - \partial_x^2 u_2(t,x) = 0 & ((t,x) \in \mathbb{R}^+ \times (0,\pi)) \\ u_2(0,x) = \sin(4x), & (x \in [0,|\pi]) \\ u_2(t,0) = u_2(t,\pi) = 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

First we solve the equation for $u_1(t,x)$. Let us search for our solution in the form $u_1(t,x) = c(t)\sin(3x)$. Then from the initial value:

$$u_1(0,x) = c(0)\sin(3x) = 0,$$

we get that $c(0) = 0$. Also, from the equation:

$$c'(t)\sin(3x) + 9c(t)\sin(3x) = \sin(t)\sin(3x)$$

$$c'(t) + 9c(t) = \sin(t)$$

This ordinary diff. equation can be easily solved, and we get

$$c(t) = ce^{-9t} + \frac{9}{82}\sin(t) - \frac{1}{82}\cos(t),$$

and because of $c(0) = 0$, $c = \frac{1}{82}$, and

$$c(t) = \frac{1}{82}e^{-9t} + \frac{9}{82}\sin(t) - \frac{1}{82}\cos(t),$$

so

$$u_1(t,x) = \left(\frac{1}{82}e^{-9t} + \frac{9}{82}\sin(t) - \frac{1}{82}\cos(t)\right)\sin(3x).$$

Now we solve the second one. Let us search for our solution in the form $u_2(t,x) = \sum_{k=1}^{\infty} \xi_k(t)\sin(kx)$ (since the eigenfunctions of the laplacian operator are $\sin(kx)$). Then from the initial value:

$$u_2(0,x) = \sum_{k=1}^{\infty} \xi_k(0)\sin(kx) = \sin(4x)$$

which means that $\xi_k \equiv 0$ if $k \neq 4$, and $\xi_4(0) = 1$. Then

$$\xi_4'(t) + 16\xi_4(t) = 0$$

from which we get that (using the initial condition) $\xi_4(t) = e^{-16t}$, and then

$$u_2(t,x) = e^{-16t}\sin(4x).$$

So the solution of the original problem is

$$u(t,x) = u_1(t,x) + u_2(t,x) = \left(\frac{1}{82}e^{-9t} + \frac{9}{82}\sin(t) - \frac{1}{82}\cos(t)\right)\sin(3x) + e^{-16t}\sin(4x).$$

7. **Let $a > 0$, and then compute the eigenvalues and the eigenvectors of the following operator!**

$$D(L) := \{u \in C^2(0,a) \cap C^1([0,a]) : u(0) = 0, u'(a) = 0\}, \qquad Lu := -5u'' + u$$

Solution: The eigenvalue-problem is

$$-5u'' + u = \lambda u$$

$$-u'' = \frac{\lambda - 1}{5}u$$

Depending on the sign of $\lambda - 1$, we have three cases:

a) If $\dfrac{\lambda-1}{5} > 0$: Then the eigenfunctions are in the form

$$u(x) = c_1 \sin\left(\sqrt{\frac{\lambda-1}{5}}x\right) + c_2 \cos\left(\sqrt{\frac{\lambda-1}{5}}x\right)$$

From the boundary conditions we have that

$$u(0) = c_2 = 0,$$

and also

$$u'(a) = c_1\sqrt{\frac{\lambda-1}{5}} \cos\left(\sqrt{\frac{\lambda-1}{5}}\, a\right) = 0.$$

If u is not the zero function, then this can only hold if

$$\sqrt{\frac{\lambda-1}{5}}\, a = \frac{\pi}{2} + k\pi$$

$$\lambda = 5\,\frac{(\pi+2k\pi)^2}{4a^2} + 1$$

So the eigenvalues are in this form (when $k \neq 0$), and the corresponding eigenfunctions are in the form

$$u_k(x) = c_2 \cos\left(\frac{\pi+2k\pi}{2a}\, x\right).$$

b) If $\dfrac{\lambda-1}{5} = 0$: Then the eigenfunctions are in the form

$$u(x) = c_1 x + c_2$$

From the boundary conditions we have that $u(0) = c_2 = 0$, and also $u'(a) = c_1 = 0$ from which $u \equiv 0$.

c) If $\dfrac{\lambda-1}{5} < 0$: Then the eigenfunctions are in the form

$$u(x) = c_1 \exp\left(\sqrt{\frac{1-\lambda}{5}}x\right) + c_2 \exp\left(-\sqrt{\frac{1-\lambda}{5}}x\right)$$

From the boundary conditions we have that

$$u(0) = c_1 + c_2 = 0,$$

so $c_1 = -c_2$ and also

$$u'(a) = c_1\sqrt{\frac{1-\lambda}{5}} \exp\left(\sqrt{\frac{1-\lambda}{5}}\, a\right) - c_2\sqrt{\frac{1-\lambda}{5}} \exp\left(-\sqrt{\frac{1-\lambda}{5}}\, a\right) =$$

$$= c_1\sqrt{\frac{1-\lambda}{5}} \left[\exp\left(\sqrt{\frac{1-\lambda}{5}}\, a\right) + \exp\left(-\sqrt{\frac{1-\lambda}{5}}\, a\right)\right] = 0$$

which can only hold if $c_1 = 0$, so $u \equiv 0$.