Exam questions

Partial Differential Equations

For all the things that are included in the fist section, only the statement of the theorems/definitions are needed, the proofs aren't. However, you should be able to prove all the statements which are mentioned in the second part.

1. Definitions, theorems

Multiindex notation. Well-posed PDE.

Linear PDE. Elliptic, hiperbolic and parabolic equation.

$C^k(\Omega)$, $C^\infty(\Omega)$, $C^k(\overline{\Omega})$. Support of a function. $C_0^k(\Omega)$. Example of a function in $C_0^k(\Omega)$. Mollifiers. Approximation theorem. Construction of some special functions. Smooth partition on unity.

The convergence $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$. Distribution. Condition equivalent to the sequential continuous property. Regular distribution. Regular distribution can correspond to one function only. Dirac-delta. Globally equivalent distributions. Locally equivalent distributions. The two distribution equivalence definitions are the same. Support of distribution.

Sum of distributions, distribution multiplied by a scalar and by a C^∞ function. $T_{\partial_j f}(\varphi) = -T_f(\partial_j \varphi)$. Partial derivative of a distribution. Derivative of a distribution. Derivative of the Heaviside function. Derivative of a regular distribution corresponding to a piece-wise differentiable function.

Cartesian product of functions. $T_{f\times g}(\varphi) = \ldots$. The Cartesian product of distributions is well-defined. Cartesian product of distributions. The Cartesian product of distributions is commutative. Linearity, derivative and support of the Cartesian product.

Convolution of functions. Sufficient condition for the existence of the convolution. $T_{f\star g} = \ldots$. Definition of the $\zeta_j \xrightarrow{(\star)} 1$ convergence. $T_{f*g} = \lim_{k\to\infty} \ldots$. Convolution of distributions. $T_{f*g} = T_f * T_g$. Identity element, commutative, linear property, support and differentiation of convolution.

Linear differential operator. Fundamental solution. Theorem of fundamental solutions.

Classical solution of the wave equation. Proposition about the weak from of the wave equation. General form of the wave equation. Classical solution is a weak solution. Existence and uniqueness of the solution of the weak equation. d'Alembert formula.

Classical solution of the heat equation. General form of the heat equation. Classical solution is a weak solution. Existence and uniqueness of the solution of the weak equation. Formula for the classical solution.

Stationary heat equation. Gauss-Ostrogradski theorem. First Green theorem. Second Green theorem. Classical boundary-value problem. Existence of the solutions of the boundary-value problem. Third-type eigenvalue problem. Theorem about operator L. The main idea of Fourier's method.

$H^k(\Omega)$. Equivalent definitions. Equivalent definitions on star-like domains. $H_0^k(\Omega)$. Equivalent definitions. Properties of $H^k(\Omega)$ and $H_0^k(\Omega)$. Equivalent norms on $H_0^k(\Omega)$. The trace operator. Equivalent definition of $H_0^k(\Omega)$ using the trace operator.

Weak form of BVPs. Connection between the classical and weak form of BVPs. The new norm is equivalent to the usual norm. The weak form of BVP has a unique solution.

2. Proofs

$\eta_{a,r} \in C_0^\infty(\Omega)$. Construction of some special functions. Smooth partition on unity.

Condition equivalent to the sequential continuous property. The two distribution equivalence definitions are the same.

$T_{f\times g}(\varphi) = \ldots$, $T_{f\star g}(\varphi) = \ldots$.

Theorem of fundamental solutions.

Proposition about the weak from of the wave equation. Existence and uniqueness of the solution of the weak equation.

First Green theorem. Second Green theorem. Existence of the solutions of the boundary-value problem. Theorem about operator L.

Equivalent norms on $H_0^k(\Omega)$. Equivalent definition of $H_0^k(\Omega)$ using the trace operator (only one direction).

Connection between the classical and weak form of BVPs. The new norm is equivalent to the usual norm. The weak form of BVP has a unique solution.