Contents

Preface

The following lecture notes were written by Bálint Takács, but most of the material is a translation of the Hungarian work of Ádám Besenyei, Vilmos Komornik and László Simon, which can be dowloaded from `http://etananyag.ttk.elte.hu/download.php?view.71`.

The document consists of two types of fonts:

- The lines written in normal sized letters are materials which would be discussed on the lectures, and they are part of the exam material.
- Lines written in smaller size are not part of the topics of the final exam, but can give an insight into the proofs which are skipped during the semester.

Since this is a 3rd year BSc (or MSc) course, there are some topics which are assumed to be known by the student - these are the following:

- Basic concepts of topology: open, closed, connected, bounded, compact sets, closure of a set, interior of a set, open ball $B(a,r)$, dense subset, distance of sets.
- Vectors and matrices: basic definitions, linear dependence, determinant, eigenvalues, scalar product, orthonormal basis, symmetric matrix, positive definite matrix, transpose, orthogonal matrix, vector space.
- Real analysis: uniformly continuous function, theorem of Heine, uniform convergence of functions.
- Differentiation of one- and multi-variable functions: partial derivatives, left and right derivatives, existence of derivatives, the set $C^2(\Omega)$, normal, gradient, divergence, Laplace operator, Young's theorem, properties of parametric integrals, Gauss-Ostrogradsky theorem, Fubini's theorem, Hölder's inequality.
- Ordinary differential equations: basic definitions, initial value, equilibrium point.
- Functional analysis: normed spaces, L^p-spaces ($1 \leq p \leq \infty$), almost everywhere convergence, convergence in norm, L^p_{loc}-spaces, linear and bounded functionals, Lebesgue's theorem, Cauchy-Schwartz inequality, operators: bounded, linear and compact, Riesz representation theorem.

If there are some of these which the student is not familiar with, we encourage him/her to refresh them from a 1st or 2nd year textbook, since the topics above will not be discussed in the lecture notes (but some of the theorems are stated).

Chapter 1

Introduction

In this course we explore the field of partial differential equations (or PDEs for short).

In the case of *ordinary differential equations* (or ODEs for short), the unknown function has only one variable, and is usually denoted by $x(t)$ (or $y(x)$, but we will stick to the former one). This type of equations can be used when we have only one variable, meaning that our unknown quantity denoted by $x(t)$ depends only on one physical variable, usually on time (this is the reason for the t notation). However, the usual processes in real life depend on not only one, but in most cases several parameters.

One of the easiest examples for this is the problem of temperature: if we ask "What is the temperature tomorrow?", it is not a good question, since we also need to specify the location. Since we already have one variable as time t ("tomorrow"), we also need another one for the location, which is usually denoted by x (e.g. Budapest). Then our function $u(t, x)$ can describe the temperature at time t and at location x. In most cases, variable t is used to describe time, and x is for space (if the process takes place in a 2 dimensional space, we can use x and y, for 3 dimensions x, y and z, and for more, e.g. n dimensions $x_1, x_2, \dots x_n$). Note that in some cases the variable x might mean not only one, but all spatial variables, meaning that $x = (x_1, x_2, \dots x_n)$ - however, in applications the most common cases are $n = 2$ and $n = 3$.

Most physical processes are described by multivariable functions, and if we would like to examine the rate of their change, we will need **partial derivatives** - this is the reason for the word "partial" in the name of PDEs.

Another difference between ODEs and PDEs is the role of initial conditions. In the case of the former ones, for a sufficiently smooth right-hand side of the equation (usually the Lipschitz property is assumed) the existence of an initial condition guarantees the existence of a unique solution of the equation. However, in the case of PDEs this is not the case, but this should be no surprise, since now we have not only one, but two or three (or even more) variables, so it is not enough to just give the value of our unknown function at $t = 0$, but we would also need some constraints for the other (spatial) variables also - the condition when the value of the function is given at some given spatial point is usually called boundary condition, since these point are usually on the boundary of our spatial set (e.g. if we examine the heat distribution in a tank, then we might know the temperature at the walls of the tank, since we can measure it there).

In the following parts of this chapter, we define some notations and basic concepts of PDEs, and then show some physical examples.

1.1 Basic concepts

1.1.1 Multiindex notation

We are going to use the notation $\partial_1, \partial_2, \dots$ for partial derivatives in the first, second, etc. variable. When we have only two (or three) variables, i.e. time t and space x (and y), then the corresponding partial derivatives are denoted by ∂_t, ∂_x, and ∂_y, respectively. Sometimes, when there are several variables in space, then the notation ∂_0 is used for the derivative in time, and $\partial_1, \partial_2, \dots \partial_n$ are used for the derivatives in space variables $x_1, x_2, \dots x_n$, respectively. We can also use higher order partial derivatives, e.g. $\partial_x \partial_y$ or $\partial_x \partial_x$, which are usually written in a shorter form as ∂_{xy} and ∂_x^2, respectively.

In this course we are going to use the **multiindex notation**, which is defined as follows. Let $\alpha_j \geq 0$, and $\alpha_j \in \mathbb{Z}$ for all $j = 1, \dots n$. Then we call the vector $\alpha := (\alpha_1, \alpha_2, \dots \alpha_n)$ a multiindex, and the "absolute value" of such multiindex is defined as $|\alpha| := \alpha_1 + \alpha_2 + \dots \alpha_n$ (if you are familiar with l^1-spaces, then this is the l^1-norm of vector α, since the values α_j are positive).

If we have a function $f : \mathbb{R}^n \to \mathbb{R}$, then

$$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f,$$

where α is defined as above, and $|\alpha|$ is the order of the derivative. This basically means that the orders of the different partial derivatives are stored in the vector α.

Example: If we have a function $f : \mathbb{R}^4 \to \mathbb{R}$ and take its partial derivative $\partial_1^2 \partial_2^3 \partial_4 f$, then $\alpha = (2, 3, 0, 1)$ ($\alpha_3 = 0$, since we do not take the derivative in that variable) and $|\alpha| = 6$, so this is a sixth order derivative.

If $|\alpha| = 0$, then our vector is in the form $(0, 0, \dots 0)$, so in this case $\partial^\alpha f = f$ (since we take no derivatives).

1.1.2 The "definition" of PDEs

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^+$) be a connected, open set. (Note that in the case when we have time $t \in [0, T]$ (so our process starts at $t = 0$ and ends at $t = T \in \mathbb{R}^+$) and also some space variable $x \in \Omega_x \subset \mathbb{R}^{n-1}$, then $\Omega = \Omega_x \times [0, T]$. Later we usually use Ω for the domain of our space variable, and omit the x from its lower-right corner, and n will be the dimension of the Ω set, not the dimension of the $\Omega \times [0, T]$ set.)

Let N be the number of $(\alpha_1, \alpha_2 \dots \alpha_n)$ multiindices for a given m, for which $|\alpha| \leq m$ (so the number of different possible types of partial derivatives is N - an upper bound for this number can be given of course, but for the sake of simplicity we are going to use this notation instead).

Let $F : \Omega \times G \to \mathbb{R}$ be an $(n + N)$-variable function (here $G \subset \mathbb{R}^N$), and the smoothness of F is often defined for a given equation.

Then a **partial differential equation** is an equation in the form

$$F(x, u(x), \partial_1 u(x), \partial_2 u(x), \dots, \partial_n u(x), \dots, \partial_n^m u(x)) = 0, \qquad (x \in \Omega) \quad (1.1)$$

or, in operator form

$$F \circ (\mathrm{id}, u, \partial_1 u, \partial_2 u, \dots, \partial_n u, \dots, \partial_n^m u) = 0, \quad (1.2)$$

in which $u \in C^m(\Omega)$ is the unknown function we would like to find. Here u is called a **classical solution** of the equation, and m is the **order** of the PDE.

The condition $u \in C^m(\Omega)$ means that u is at least m times continuously differentiable, which is a natural requirement, since the equation has $\partial_n^m u$ in it - however, usually the existence of all derivatives is not required (in equation (1.1) the non-present terms can be thought of as being multiplied by zero), and in these cases less strict conditions are given. Also, later in the semester we will define **weak solutions** of the equations, for which the existence of derivatives is not required, but other conditions are needed (which result in a less smooth function).

1.1.3 Initial and boundary conditions, well-posedness

Just like in the case of ODEs, the solution of a PDE usually means a set of functions. However, while in the case of the former these solutions usually differ from each other only by a constant, the solutions of a PDE can be totally different depending on the initial and the boundary conditions. Here we are going to suppose that we have a time variable $t \in \mathbb{R}$ (usually $t \in [0, T]$, $T \in \mathbb{R}^+$) and a space variable $x \in \Omega \subset \mathbb{R}^n$ (usually Ω is a bounded, connected and open set).

I. **Boundary conditions:** These mean that the value of u or the value of ∂u (or the directional derivative of u - more on this later) is given on the boundary of our domain Ω. These problems are called **boundary-value problems** (or BVPs for short).

   For example, if our domain is $\Omega = [0, 1] \subset \mathbb{R}$ (so this is a one-dimensional problem), then a boundary condition means that the value of $u(0, t)$ or $u(1, t)$ (or $\partial_x u(0, t)$) is given.

   If we do not have a time variable (this means that our solution is constant in time), then we only need boundary conditions.

II. **Initial condition:** This means that the value of u or ∂u is given at some time t, in most cases $t = 0$. These problems are called **initial-value problems** (IVPs) or **Cauchy-problems**.

   For example, the value of $u(x, 0)$ (or $\partial_t u(x, 0)$) is given.

   Usually initial-value problems arise when the partial differential equations are thought of as ordinary differential equations defined on some abstract space (usually Hilbert spaces), and then they are called abstract Cauchy-problems (ACP). These concepts are used in the theory of operator semigroups.

III. **Initial-boundary value problems** In most cases, both initial and boundary conditions are needed to ensure that our equation has a unique solution. These problems are called **initial-boundary value problems** (or IBVPs for short).

In applications, it is important that the equation we write up has some nice properties: an equation having two solutions is usually not a physically reasonable one (an object will not move in two different ways). The next definition collects the most important required properties.

Definition 1.1. A PDE is said to be **well-posed**, if the following three conditions hold:

(i) **Existence:** A solution exists in the observed class of functions (in the case of classical solutions, this class is $C^m(\Omega)$).

(ii) **Uniqueness:** There is at most one solution in that class of functions.

(iii) **Stability/continuous dependence on parameters:** Depending on the textbooks, this condition might mean two different things:

- If we modify the initial or boundary conditions a little, then the solution of this new equation should be close to the original one.
- A small change of the equation (e.g. the change of a constant or a given function) results in a small change of the solution.

Requirements (i) and (ii) seem to be rather natural, but condition (iii) is also very important in applications: usually the given parameters or the initial/boundary conditions come from physical measurements, which are only precise up to some extent. Then, it is a natural requirement that if our measurements are precise up to e.g. 4 digits, then our solution should also be precise up to 4 digits.

It is also a natural requirement that a small change in parameters should result in a slightly different solution and not a totally different one. This is important because of the aforementioned problem of precision: if one of the constants is measured to be 1.000 but the equation has two totally different solutions in the cases 0.9999 and 1.0001, then this is an issue since we do not know which of these two different solutions is the 'real one'. Unfortunately, in the second half of the 20th century several systems were discovered in which this property does not hold - these are then said to be *sensitive to the initial conditions*. Such systems include not only large, complex systems like the weather, but also rather simple ones, like the double pendulum. The branch of mathematics which deals with such systems is called *chaos theory*, but it is not discussed in this course.

1.2 Physical examples

In this section some physical examples are presented which can be modeled by using PDEs.

1.2.1 Thermal conductivity

Thermal conductivity means the passing of heat through some material. The main physical idea here is called Fourier's law, which basically means that "the warm air flows to the places where it is cold". (So when grandma says "Please close the window because the cold air comes in!", she is wrong, since it is the warm air that goes out of the window.)

If we would like to be more precise, Fourier's law states that if we fix a piece of a plane at some point x with area δA, then the amount of heat δQ passing through this piece of plane in δt time is

$$\delta Q = -k(x)\, \partial_\nu u(x,t)\, \delta A\, \delta t. \tag{1.3}$$

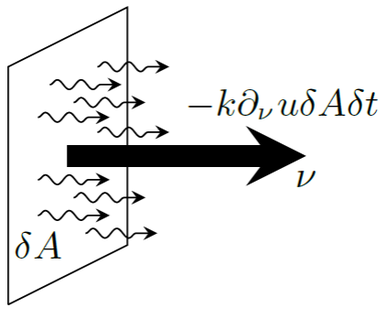


Figure 1.1: The law of Fourier[1].

(See Figure 1.1.) Here:

- ν is the normal of the plane, and

$$\partial_\nu u(x,t) = (\mathrm{grad}(u)) \cdot \nu,$$

  which is called the directional derivative in that direction of the normal, or **normal derivative** in short. (Here $\cdot$ means scalar product of vectors.)

- $k(x)$ is the termal conductivity (a physical constant which comes from the properties of the material the heat moves in).

- $u(t,x)$ is the temperature at time t and at (spatial) point x.

- The negative sign in front of the right-hand side comes from the physical meaning of the expression: if $\partial_\nu u > 0$, then it means that the heat moves out of the domain, but then the amount of heat should decrease, so since all of the constants are positive, $\delta Q < 0$.

In the next pages we will try to get a PDE from the physical law above - first, we do it in one dimension (i.e. when $x \in \Omega \subset \mathbb{R}$), and then only mention the main ideas of the multidimensional case.

a) **One dimensional case:** Let us suppose that we have a thin rod with length $L \in \mathbb{R}^+$, and its surface is insulated meaning that no heat can escape from it. Since it is very thin, its points can be modeled as $x \in [0, L]$, since it is assumed that there is no difference in its temperature at points at a given length. Then let us denote by $u(x,t)$ the temperature of the rod at time t and at point x .

Let us consider a small part of the rod, namely the section $[x, x+\delta x]$ (where $\delta x \in \mathbb{R}^+$ is a small value). Then this is a cylinder with height δx, and let us suppose that the base of this cylinder has the area δA. Then by Fourier's law, the amount of heat passing through in time δt at the base at point x is $(k\partial_\nu u)|_{(x,t)} \delta A\ \delta t$, and the heat escaping at the other end is $-(k\partial_\nu u)|_{(x+\delta x,t)} \delta A\ \delta t$ (see Figure 1.2).

Then the difference of heat in the rod is

$$\delta Q_1 = \left[k(x+\delta x)\partial_x u(x+\delta x,t) - k(x)\partial_x u(x,t)\right] \delta A\ \delta t \approx$$

$$\approx \partial_x \left(k(x)\partial_x u(x,t)\right) \delta x\ \delta A\ \delta t$$

(Here the term δx comes from the definition of partial derivatives.)

[1]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

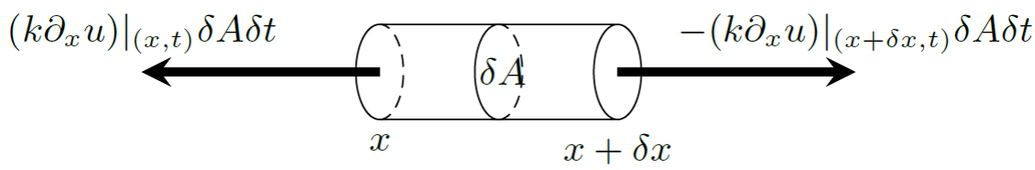


Figure 1.2: Heat conductivity in a small section of the thin rod[2].

Let $F(x,t)$ be some outer source or sink of heat (we warm up, or cool down our rod at some points). Then, the difference of heat coming from this procedure on this small section in time δt is

$$\delta Q_2 = F(x,t)\ \delta x\ \delta A\ \delta t.$$

Also, the change of temperature in time δt at point x can be approximated by the derivative of u in variable t:

$$u(x,t+\delta t) - u(x,t) \approx \partial_t u(x,t)\delta t$$

We know from physics that the amount of heat needed for such warming/cooling is

$$\delta Q_3 \approx c(x)\ \rho(x)\ \delta x\ \delta A\ \partial_t u(x,t)\ \delta t$$

in which $c(x)$ is the heat capacity, and $\rho(x)$ is the density of the material (both are physical constants).

Since the total amount of change of heat is δQ_3, and this change can only come from either the move of the heat (which was described by δQ_1) or from the outer heating/cooling (denoted by δQ_2), then

$$\delta Q_3 = \delta Q_1 + \delta Q_2$$

$$c(x)\ \rho(x)\ \delta x\ \delta A\ \partial_t u(x,t)\ \delta t = \partial_x\left(k(x)\partial_x u(x,t)\right)\delta x\delta A\delta t + F(x,t)\delta x\delta A\delta t$$

If we divide this by δt, δA and δx, we get:

$$c(x)\ \rho(x)\ \partial_t u(x,t) - \partial_x\left(k(x)\partial_x u(x,t)\right) + F(x,t)$$

When $c(x)$, $\rho(x)$ and $k(x)$ are constants (meaning that they do not depend on x), then we get

$$\partial_t u(x,t) - \frac{k}{c\rho}\partial_x^2 u(x,t) = \frac{F(x,t)}{c\rho},$$

We can also introduce a new constant $a = \dfrac{k}{c\rho}$ (which is sometimes called the *thermal diffusivity constant*), and also define $f(x,t) = \dfrac{F(x,t)}{c\rho}$.

(Note that all of the arguments above can be made rigorous, but here we only wanted to show the main ideas of the proof.)

In the end, we get the following equation:

$$\partial_t u(x,t) - a\ \partial_x^2 u(x,t) = f(x,t) \tag{1.4}$$

which is the **inhomogeneous one dimensional heat equation** (here a is a positive constant and $f(x,t)$ is a given function). The term $f(x,t)$ is usually called the **source term**, since it describes the effect of an outer source or sink of heat - when $f(x,t) \equiv 0$, then the equation is called the **homogeneous heat equation**.

As mentioned before, for our equation to have a unique solution, we will need some constraints.

[2]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

1. **Initial condition:** Let us suppose that we know the initial temperature of our rod at all points, i.e.

$$u(x,0) = T_0(x), \qquad x \in [0,L],$$

in which $T_0(x)$ is a known function.

For our boundary condition, we have three choices:

2. **First-type boundary condition:** In this case the value of function $u(x,t)$ is known at the endpoints of our domain $[0,L]$:

$$\begin{aligned} u(0,t) &= T_1(t), \qquad (t \geq 0) \\ u(L,t) &= T_2(t). \qquad (t \geq 0) \end{aligned}$$

in which $T_1(t)$ and $T_2(t)$ are known functions (e.g. we have a thermometer installed at both ends of the rod). This type of boundary condition is called **first-type** or **Dirichlet boundary condition**.

3. **Second-type boundary condition:** In this case the value of $\partial_x u(x,t)$ is known at the endpoints of our domain $[0,L]$:

$$\begin{aligned} k(0)\partial_x u(0,t) &= u_1(t), \qquad (t \geq 0) \\ k(L)\partial_x u(L,t) &= u_2(t). \qquad (t \geq 0) \end{aligned}$$

in which $u_1(t)$ and $u_2(t)$ are known functions. This type of boundary condition is called **second-type** or **Neumann**[3] **boundary condition**. Note that since $x \in [0,L]$, the partial derivatives taken at points $x = 0$ and $x = L$ are meant to be the right and left derivatives of the function taken at those points, respectively.

4. **Third-type boundary condition:** In this case the values of both $u(x,t)$ and $\partial_x u(x,t)$ are known at the endpoints of our domain $[0,L]$:

$$\begin{aligned} k(0)\partial_x u(0,t) - \lambda u(0,t) &= -\lambda T_k(t), \qquad (t \geq 0) \\ k(L)\partial_x u(L,t) + \lambda u(L,t) &= \lambda T_k(t). \qquad (t \geq 0) \end{aligned}$$

in which λ is a constant, and $T_k(t)$ is a known function. This type of boundary condition is called **third-type** or **Robin boundary condition**.

Our PDE is defined on the set $Q := (0,L) \times \mathbb{R}^+$ (see Figure 1.3). If our rod is infinite ($x \in \mathbb{R}$), then only the initial condition is needed, and then we have an initial-value problem.

Another question is the function space in which we would like to search for our solution: this could either be $C^2(\overline{Q})$, $C^2(Q) \times C^1(\overline{Q})$ or just $C^1(Q) \times C(\overline{Q})$ in which case we also assume that $\partial_x^2 u \in C(\overline{Q})$. This decision will be made later.

[3]This is named after Carl Neumann, who is famous for the Neumann series (see your Functional Analysis class for those), and not after the "other" famous Neumann, John von Neumann.

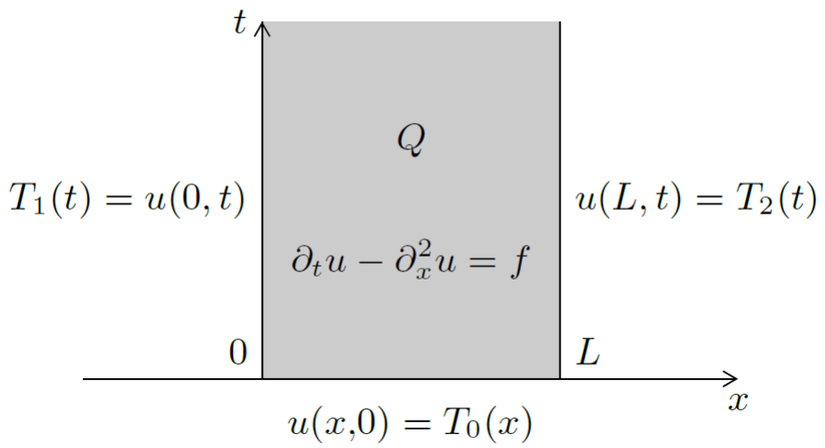


Figure 1.3: The set the heat equation is defined on with the initial and boundary conditions[5].

b) **More dimensions:** By similar arguments as before, we can consider a plane (a very thin iron plate) and a small rectangle on it (see Figure 1.4).

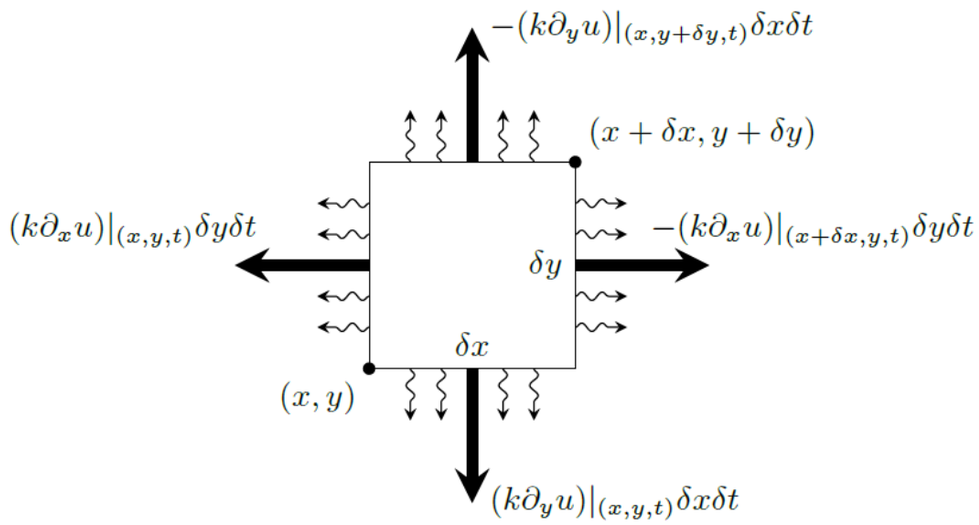


Figure 1.4: Heat conductivity in a small rectangle[6].

After a proof similar to the one presented in the one-dimensional case, we get the equation

$$\partial_t u(x,y,t) - \frac{k}{c\rho}\Delta u(x,y,t) = \frac{F(x,y,t)}{c\rho} \tag{1.5}$$

or

$$\partial_t u(x,y,t) - a\Delta u(x,y,t) = f(x,y,t)$$

with notations $a = \dfrac{k}{c\rho}$ and $f(x,y,t) = \dfrac{F(x,y,t)}{c\rho}$. When k is not a constant function, we get

$$\partial_t u(x,y,t) - \frac{1}{c\rho}\mathrm{div}\,(k(x,y)\mathrm{grad}(u(x,y,t))) = \frac{F(x,y,t)}{c\rho}. \tag{1.6}$$

[5]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.
[6]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

Note that the Laplace, the div and the grad operators present in the previous formulas are acting only on the space variables, and not on the time one, i.e.

$$\Delta u(x,y,t) = \partial_x^2 u(x,y,t) + \partial_y^2 u(x,y,t),$$

and in the three dimensional case (where $x = (x_1, x_2, x_3)$)

$$\Delta u(x,t) = \sum_{j=1}^{3} \partial_j^2 u(x,t).$$

We can also define an initial condition for this problem, namely

$$u(x,y,0) = \varphi(x,y), \qquad ((x,y) \in \overline{\Omega})$$

in which $\varphi : \overline{\Omega} \to \mathbb{R}$ is a given function ($\Omega \subset \mathbb{R}^2$).

The boundary conditions are similar to the one dimensional case:

- **First-type (or Dirichlet) boundary condition:** The value of u is given at the boundary of the domain.
$$u|_{\partial\Omega} = \chi$$
in which $\chi : \partial\Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ is a given function (here $\partial\Omega$ means the boundary of the set Ω).
- **Second-type (or Neumann) boundary condition:** The normal derivative of $u(x,y,t)$ is given at the boundary of the domain.
$$\partial_\nu u|_{\partial\Omega} = \chi$$
in which $\chi : \partial\Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ is a given function.
- **Third-type (or Robin) boundary condition:** A linear combination of the value and the normal derivative of $u(x,y,t)$ is given at the boundary of the domain.
$$\alpha\, \partial_\nu u|_{\partial\Omega} + \beta\, u|_{\partial\Omega} = \chi$$
in which $\alpha, \beta, \chi : \partial\Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ are a given functions.

In this case our equation is defined on the set $Q := \Omega \times \mathbb{R}^+$ (since $x \in \Omega \subset \mathbb{R}^2$ and $t \in \mathbb{R}^+$). When $\Omega \subset \mathbb{R}^2$, this domain looks like a cylinder (see Figure 1.5), while it is a bit more difficult to visualize it when $\Omega \subset \mathbb{R}^n$, $n > 2$. Note that when $n = 1$, we get back the same picture as in the one dimensional case (see Figure 1.3).

Remark 1.1. When we consider some other physical processes, e.g. the diffusion of gases, we get a similar equation as (1.4) or (1.5). In this case u denotes the density of the material, and instead of Fourier's law, we use **Fick's law**, which is very similar to the former one: the gas moves to those places where the density is lower. This is the reason why equation (1.4) or (1.5) is sometimes called **diffusion equation**.

Remark 1.2. Note that if we use the physical concept of Brownian motion and the tools of stochastic analysis, we will get the same equation as (1.4) or (1.5).

c) **Stationary conductivity:** In some cases we would like to observe the states of the heating process which do not change in time, which means that $\partial_t u = 0$. Such solutions are important since they are usually the

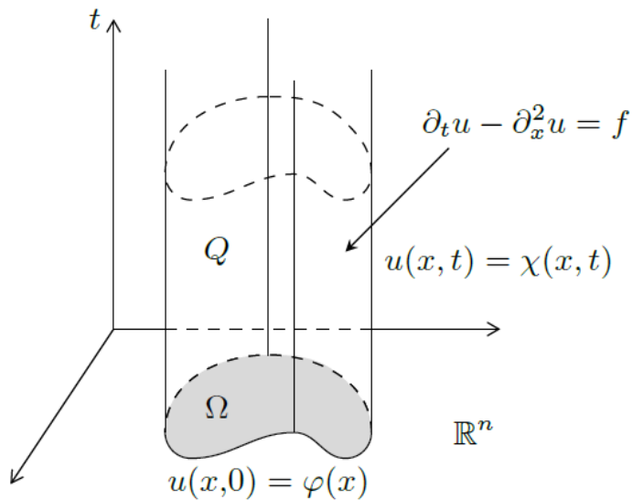


Figure 1.5: The set our equation is defined on[7].

limits of other solutions: this is a similar concept as equilibrium points in the case of ordinary differential equations - in the case of PDEs, these are called *equilibrium solutions*.

When $\partial_t u = 0$, then equation (1.6) has the form

$$-\operatorname{div}(k(x,y)\operatorname{grad}(u(x,y,t))) = F(x,y,t). \qquad ((x,y) \in \Omega) \tag{1.7}$$

When $k(x,y) \equiv 1$, we get

$$-\Delta u = F. \tag{1.8}$$

Equation (1.8) is usually called **Poisson equation**. A special case of this one is when $F = 0$, which results in

$$-\Delta u = 0. \tag{1.9}$$

Equation (1.9) is usually referred to as the **Laplace equation**. Boundary conditions can be defined similarly as in the previous cases.

1.2.2 Wave equation

On the following pages we introduce the wave equation, which can be used to model the movement of a string, or the top of a drum. It is important to emphasize that it cannot be used to model the movement of waves of the ocean: for that other kind of equations (shallow-water equations) are needed.

a) **One dimension:** Let us consider a rope with constant length L, and assume that it moves only vertically. Let us model the rope in a two-dimensional coordinate system: let us use choose a given height to be $y = 0$, and use variable x for the horizontal distance from the origin of this coordinate system. Then, let us denote by $u(x,t)$ the distance of a given point of the rope from the x-axis at time t: $u(x,t) > 0$ when the rope is above this axis, and $u(x,t) < 0$ when the rope is below it (see Figure 1.6).

Let the tangential force acting on the rope be denoted by T, and μ be the linear density of the rope (which means that its mass m can be calculated as $m = \mu \cdot L$).

[7]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

The outer force acting on a small segment $[x, x+\delta x]$ is $F \cdot \delta x$ (e.g. there is a wind that moves the rope). Let us consider this small segment and observe the forces that act on it. There is $F\delta x$ which comes from outer effects, and T comes from the rope itself (the force that tries to hold the rope together). They can be seen on Figure 1.6.

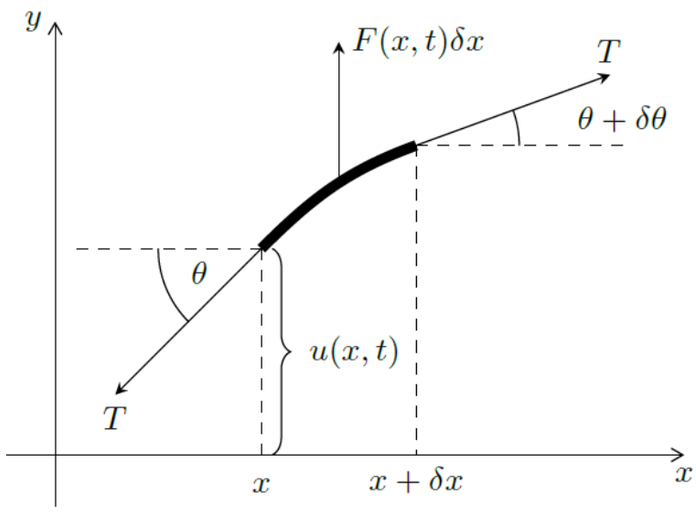


Figure 1.6: The forces acting on a small segment of the rope[8].

Now we use the second law of Newton, which states that the sum of all forces $\sum F$ acting on an object equals to the product of its mass m and the acceleration a that was caused by the forces:

$$\sum F = m \cdot a. \tag{1.10}$$

The force on the left end of the segment is $-T\sin(\theta)$, while the one on the right is $T\sin(\theta+\delta\theta)$ (the reason for the sin is that we are going to consider the vertical component of force T, since this is the direction the rope moves, so the vector of acceleration points into this direction). Then, if we substitute into equation (1.10), we get

$$T\sin(\theta+\delta\theta) - T\sin(\theta) + F(x,t)\ \delta x = (\mu\ \delta x)\ a_y, \tag{1.11}$$

in which the term $\mu\ \delta x$ is the mass of the rope, and a_y is the acceleration taken in the vertical (y) direction (since all the forces are taken in that one).

Then, we can use some approximations: for small values of $\delta\theta$, $\sin(\delta\theta) \approx \delta\theta$ and $\cos(\delta\theta) \approx 1$, which means that

$$\sin(\theta+\delta\theta) = \sin(\theta)\cos(\delta\theta) + \cos(\theta)\sin(\delta\theta) \approx \sin(\theta) + \cos(\theta)(\delta\theta).$$

When θ is small, $\sin(\theta) \approx 0$ and $\cos(\theta) \approx 1$, so (1.11) becomes (for small values of θ and $\delta\theta$)

$$T(\delta\theta) + F(x,t)\ \delta x = (\mu\ \delta x)\ \partial_t^2 u(x,t), \tag{1.12}$$

in which we used that $a_y = \partial_t^2 u(x,t)$, since acceleration is the second derivative taken with respect to the time variable.

Now we use the approximation

$$\tan(\theta) \approx \frac{\delta u}{\delta x} \approx \partial_x u.$$

If we differentiate the expression above with respect to variable x, we get

$$\frac{1}{\cos^2\theta}\partial_x\theta = \partial_x^2 u.$$

For small values of θ, $\cos^2\theta \approx 1$, and $\partial_x\theta \approx \dfrac{\delta\theta}{\delta x}$, meaning that

$$\delta\theta \approx \partial_x^2 u \cdot \delta x.$$

[8]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

If we use this approximation in equation (1.12), we get

$$(T\delta x)\partial_x^2 u + F(x,t)\ \delta x = (\mu\ \delta x)\ \partial_t^2 u(x,t). \tag{1.13}$$

Then, by dividing it by δx and then by μ, we get

$$\partial_t^2 u(x,t) - \frac{T}{\mu}\partial_x^2 u(x,t) = \frac{F(x,t)}{\mu}. \tag{1.14}$$

Let us use the notation $f := \dfrac{F}{\mu}$ and $c^2 = \frac{T}{\mu}$.

Then, after some calculations we get the equation

$$\partial_t^2 u(x,t) - c^2\partial_x^2 u(x,t) = f(x,t), \tag{1.15}$$

in which c is a positive constant and $f(x,t)$ is a given function (usually called the source term). (1.15) is called the **one dimensional wave equation**. When $f(x,t) = 0$, we call it **homogeneous**, while the case $f(x,t) \neq 0$ is called **inhomogeneous**.

As in the case of the heat equation, we would also need some constraints that guarantee the existence of a unique solution.

1. **Initial conditions:** Since we have second derivatives in time, we would need two conditions for our solution to be unique (similarly as in the case of second order ordinary differential equations):

$$\begin{aligned} u(x,0) &= \varphi(x), \\ \partial_t u(x,0) &= \psi(x), \end{aligned}$$

   in which $\varphi(x)$ and $\psi(x)$ are two given functions. Here $\partial_t u(x,0)$ means the initial velocity distribution of the rope.

We have several choices for the boundary conditions depending on the movement of the endpoints of the rope:

2. **Ends with given motion:** Let us suppose that we know the motion of our endpoints during the process, e.g. we are the ones moving them. Then the boundary conditions are:

$$\begin{aligned} u(0,t) &= \chi_1(t), \\ u(L,t) &= \chi_2(t). \end{aligned}$$

   Here $\chi_1(t)$ and $\chi_2(t)$ are given functions, and L is the length of the rope. In this case this is a **Dirichlet condition** (since the value of the unknown function is given). Note that when $\chi_1 = \chi_2 = 0$ (or when they are constant), the endpoints are fixed.

3. **Free endpoints:** Let us suppose that the endpoints are free to move, e.g. they are tied to a pole, and can move up or down without any friction (see Figure 1.7). Then it can be shown that the boundary conditions in this case have the form

$$\begin{aligned} \partial_x u(0,t) &= 0, \\ -\partial_x u(L,t) &= 0. \end{aligned}$$

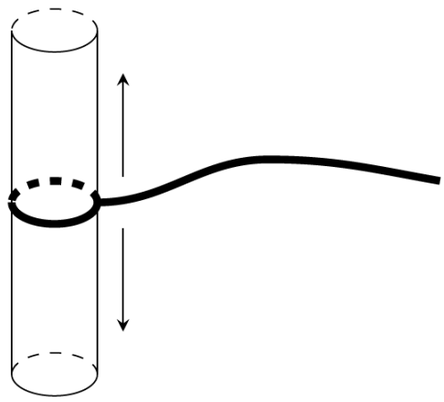


Figure 1.7: The case when the endpoint of the rope can move freely[9].

As we can see, this is a **Neumann boundary condition**. Here $\partial_x u(0,t)$ is the right, while $\partial_x u(L,t)$ is the left derivative of the function calculated at points $x = 0$ and $x = L$, respectively.

4. **Endpoints connected to a string:** In this case the endpoint of the rope is connected to a pair of strings, so it can move, but the movement is slowed down by the strings (see Figure 1.8). It can be shown that the boundary conditions have the form

$$\alpha_1 \partial_x u(0,t) + \beta_1 u(0,t) = \chi_1(t),$$
$$\alpha_2 \partial_x u(L,t) + \beta_2 u(L,t) = \chi_2(t),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are known constants, and χ_1, χ_2 are given functions. Also, this is a **Robin-type boundary condition**.

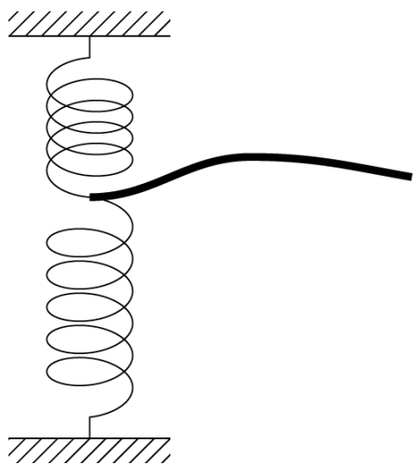


Figure 1.8: The case when the endpoint of the rope is connected to a pair of strings[10].

In the case of the wave equation, the space our equation is defined on is $Q := (0, L) \times \mathbb{R}^+$ (since $x \in (0, L)$ and $t \in \mathbb{R}^+$). In some cases it is assumed that the rope is infinitely long, so then we do not need boundary conditions, and our space is $Q := \mathbb{R} \times \mathbb{R}^+$ ($x \in \mathbb{R}$).

[9]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.
[10]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

b) **Multidimensional case:** Let us assume that we have a flexible thin plane (e.g. a blanket), and we would like to model its movement - in this case $x \in \Omega \subset \mathbb{R}^2$, or if we consider more than two-dimensional waves, $\Omega \subset \mathbb{R}^n$. If we use the same ideas as in the one dimensional case, we get a similar equation:

$$\partial_t^2 u - c^2 \Delta u = f. \tag{1.16}$$

Here the Laplace operator acts only on the space variables $x_1, x_2, \dots x_n$, namely

$$\Delta u(x,t) = \sum_{j=1}^{n} \partial_j^2 u(x,t).$$

The initial and boundary conditions are also similar to the ones in the one dimensional case.

1.2.3 Other examples

In this section we show some further, widely used physical processes which can be modeled using PDEs.

1. **Transport equation:** Suppose that we have a pipe in which some fluid is moving, containing some material. Then let us denote by $u(x,t)$ the density of the transported material at position x in the pipe at time t, and let $v(x,t)$ be the speed of the current at point x and at time t. It is also possible to model the movement of fluids in 2 dimensions, e.g. the movement of some pollution in the ocean transported by the currents.

   The equation describing this movement is

   $$\partial_t u(x,t) + v(x,t) \cdot \operatorname{grad}(u(x,t)) = 0 \tag{1.17}$$

   in which $\cdot$ is a scalar product, and the gradient operator only acts on the space variables. Equation (1.17) is called the **transport equation**. It is also possible to add a source term f to the right-hand side of the equation, meaning a source or sink of the material.

2. **Biharmonic equation:** In most applications, we consider second order equations, because we would like to model something that moves because of diffusion (Laplace operator), or in a wave-like motion (second derivative in time). An interesting exception is the **biharmonic equation**, which is used to model linear elasticity, and has the form

   $$\Delta^2 u = 0, \tag{1.18}$$

   in which the *biharmonic operator* Δ^2 is defined as

   $$\Delta^2 u = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i^2 \partial_j^2 u.$$

   Remark 1.3. An interesting fact is that in recent years the non-integer "powers" of the Laplace operator were also defined to model diffusion more accurately, but the definition of such operators is far beyond the scope of this course.

3. **Systems of PDEs:** In most applications we do not have only one unknown function, but several ones, meaning that we need multiple equations to model them. Here we mention only two examples of such systems:

   - **Maxwell equations**, which are used in electro-magnetic processes.
   - **Navier Stokes equations**, which are used to model currents and hydrodinamics - these are the ones which can be used to model the movement of waves of the ocean. Although they are widely used, there are still some open questions regarding them - you can even get a million dollars if you can answer these questions (see `https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_existence_and_smoothness` for details).

In the next chapter we start to examine second order linear equations, which will be the main topic of the further parts of the course.

Chapter 2

Second order linear PDEs

In this chapter we discuss the classification of second order linear partial differential equations, and then show a useful transformation which can be used to transform a wide class of such equations into a more simple form (called the canonical form).

2.1 General form of second order linear PDEs

Definition 2.1. We say that a second order partial differential equation is **linear in its main term**, if it is in the form

$$\sum_{j,k=1}^{n} a_{j,k}\partial_j\partial_k u = g \circ (\mathrm{id}, u, \partial_1 u, \dots \partial_n u)\,, \tag{2.1}$$

in which $a_{j,k} : \Omega \to \mathbb{R}$ and $g : \Omega \times G \to \mathbb{R}$ are fixed functions, while $G \subset \mathbb{R}^{n+1}$ is a fixed interval[1].

This basically means that the second order terms (the ones in the form $\partial_j\partial_k u$) form a linear combination, so that part of the equation (the main part) is linear, while the right-hand side might contain any nonlinear function.

Definition 2.2. We say that an equation is a **second order linear partial differential equation**, if it is in the form

$$\sum_{j,k=1}^{n} a_{j,k}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu = f, \tag{2.2}$$

in which $a_{j,k}, b_j, c, f : \Omega \to \mathbb{R}$ are given functions.

If we would like to get a classical solution of our problem, we would need that $u \in C^2(\Omega)$. However, by the theorem of Young, we know that in this case $\partial_j\partial_k u = \partial_k\partial_j u$ for all j, k indices. Because of this, we can regroup the terms $a_{j,k}\partial_j\partial_k u$ and $a_{k,j}\partial_k\partial_j u$ in a way that $a_{j,k} = a_{k,j}$. Because of this, we can make the following assumption.

Assumption 1. For all j, k indices, $a_{j,k} = a_{k,j}$.

Remark 2.1. In the case of third order equations, a similar assumption can be made, since in that case $a_{j,k,l} = a_{j,l,k} = a_{k,j,l} = \dots$, which comes from an extended form of Young's theorem (which can be proved using the original Young theorem and induction).

[1] In this context **interval** means an open subset of $\mathbb{R}^{n+1}$ in the form $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_{n+1}, b_{n+1}]$ where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$ for all $i = 1, 2, \dots n, n+1$.

2.2 Classification

In this section we classify the second order linear equations depending on their main term.

Let $x_0 \in \Omega \subset \mathbb{R}^n$ be a fixed point in our domain (which now might also contain the time variable), and let $A(x_0)$ be the matrix constructed from the coefficients of the main terms in equations (2.1) and (2.2), meaning that:

$$A = A(x_0) := [a_{j,k}(x_0)]_{j,k=1}^n = \begin{pmatrix} a_{1,1}(x_0) & a_{1,2}(x_0) & \dots & a_{1,n}(x_0) \\ a_{2,1}(x_0) & a_{2,2}(x_0) & \dots & a_{2,n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x_0) & a_{n,2}(x_0) & \dots & a_{n,n}(x_0) \end{pmatrix}. \quad (2.3)$$

Since $a_{j,k}(x_0) = a_{k,j}(x_0)$ by Assumption 1, we get that the matrix $A(x_0)$ is symmetric. Now we are going to use the following theorem coming from Linear Algebra (which is not proved here, but its proof can be found in many Linear Algebra textbooks):

Theorem 2.1. *Let us consider a matrix A with real entries, and suppose that the matrix is symmetric. Then all of the eigenvalues of the matrix are real, and the eigenvectors form an orthonormal basis in $\mathbb{R}^n$.*

By Theorem 2.1, we know that our matrix A defined in (2.3) has real eigenvalues, and its vectors form an orthonormal basis.

Let us introduce the following notations:

- n_+ : the number of positive eigenvalues of A (counted with (algebraic) multiplicity, meaning that if we have only one positive eigenvalue but with (algebraic) multiplicity 2, then $n_+ = 2$).
- n_- : the number of negative eigenvalues of A (counted with (algebraic) multiplicity).
- n_0 : the number of zero eigenvalues of A (counted with (algebraic) multiplicity).

Note that $n_+ + n_- + n_0 = n$.

Now we state the main definition of this section:

Definition 2.3. Equation (2.1) and (2.2) **at point $\mathbf{x_0}$** is called:

- **elliptic**, if $n_+ = n$ or $n_- = n$, so all of the eigenvalues have the same sign.
- **hyperbolic**, if $n_+ = 1$ and $n_- = n - 1$, or $n_- = 1$ and $n_+ = n - 1$, so all of the eigenvalues have the same sign except one.

  If $n_0 = 0$ and $n_+ > 0$ and $n_- > 0$, the equation is sometimes called **ultrahyperbolic**.
- **parabolic**, if $n_0 = 1$ and $n_+ = n - 1$, or $n_0 = 1$ and $n_- = n - 1$, so all of the eigenvalues have the same sign except one, which is zero.

  Sometimes the equation which has at least one zero eigenvalue is called **parabolic in the broader sense**.

We say that an **equation is elliptic/hyperbolic/parabolic**, if it is elliptic/hyperbolic/parabolic at all of the points $x_0 \in \Omega$.

Remark 2.2. Note that the above names correspond to the definitions of hyperbolic, parabolic and elliptic curves in geometry (the distribution of the signs of the eigenvalues are the same in the corresponding matrices). For example, a second order curve given in the form $x^T M x = 0$ is called *elliptic*, if the eigenvalues of the matrix M have the same sign.

The definition of an elliptic equation can be reformulated in the following way: if all of the eigenvalues are positive, then the matrix is positive definite, which means that

$$\langle A(x_0)p, p\rangle \geq \min_{j=1,\dots n} \left(\lambda_j(x_0)\right) |p|^2,$$

for all $p \in \mathbb{R}^n$ vectors, in which $\lambda_j(x_0)$ denotes the eigenvalue of $A(x_0)$. (Here $\langle\ .\ ,\ .\ \rangle$ is the usual scalar product defined for vectors.) However, the right-hand side can be an arbitrary small positive number: we are introducing a more strict definition which requires a positive lower bound.

Definition 2.4. We say that our equation is **uniformly elliptic at some set $\Omega \subset \mathbb{R}^n$**, if there exists $c_0 > 0$ and $c_1 > 0$ constants for which

$$c_1|p|^2 \geq \langle A(x_0)p, p\rangle \geq c_0|p|^2$$

for all $p \in \mathbb{R}^n$ and for all $x_0 \in \Omega$.

Corollary 2.2. *If an equation is uniformly elliptic inside a set Ω, then it is also elliptic inside that set.*

Proof. If an equation is uniformly elliptic inside set Ω, then $\langle A(x_0)p, p\rangle \geq c_0|p|^2 > 0$ for every $x_0 \in \Omega$, which means that A is positive definite by definition, so all of its eigenvalues are positive, meaning that it is elliptic at every point $x_0 \in \Omega$. □

Remark 2.3. It is not hard to see (using spectral theorem) that the uniform elliptic property means that the eigenvalues of the matrix are positive, but are bounded from below (and also from above) by a positive constant. If an operator is elliptic but not uniformly, it means that its eigenvalues might get arbitrary close to zero (see the lines before the previous definition).

Now we consider some equations, and categorize them based on the previous definitions.

Examples:

1. **n-dimensional Laplace equation:** It has the form

$$\Delta u = \sum_{j=1}^{n} \partial_j^2 u = 0.$$

Then, the matrix which can be constructed from its second order terms (which means all of the terms in this case) has the form of an identity matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Its eigenvalues are positive (+1), so the equation is *elliptic* (since $n_+ = n$).

Also, since

$$\langle Ap, p\rangle = \langle p, p\rangle = \sum_{j=1}^{n} p_j^2 \leq |p|^2$$

and

$$\langle Ap, p\rangle \geq \min_{j=1,\ldots n} \left(\lambda_j(x_0)\right) |p|^2 = |p|^2,$$

then it is also uniformly elliptic (with $c_0 = c_1 = 1$).

2. **n-dimensional wave equation:** It has the form

$$\partial_t^2 u - \Delta u = \partial_t^2 u - \sum_{j=1}^{n} \partial_j^2 u = 0.$$

Then the matrix which can be constructed from its second order terms (which means all of the terms in this case) has the following form:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix},$$

so all of its eigenvalues are +1 except one, which is −1: this means that the equation is *hyperbolic* ($n_+ = 1$, $n_- = n - 1$).

3. **n-dimensional heat equation:** It has the form

$$\partial_t u - \Delta u = \partial_t u - \sum_{j=1}^{n} \partial_j^2 u = 0.$$

Since it does not contain any second order derivatives with respect to variable t, the coefficient corresponding to that term is zero: thus, the matrix we get is

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix},$$

which has one zero eigenvalue, all of the others are −1, so it is *parabolic* ($n_0 = 1, n_- = n - 1$).

2.3 Canonical form (skipped in 2024)

The following subsections are skipped in 2024.

In this section we introduce a transformation which can be used to simplify our second order equations into a form which is easier to handle.

2.3.1 Semilinear equation with constant coefficients

Let us suppose that our equation is in the form

$$\sum_{j,k=1}^{n} a_{j,k}\, \partial_j \partial_k u = g \circ (\mathrm{id}, u, \partial_1 u, \dots, \partial_n u)\,, \tag{2.4}$$

which is defined on the set $\Omega \subset \mathbb{R}^n$, $a_{j,k} \in \mathbb{R}$ are constants, and Assumption 1 holds, namely $a_{j,k} = a_{k,j}$ for all j, k indexes.

Let $\lambda_1, \dots \lambda_n$ be the eigenvalues of A (which is constructed as in the previous section), and the corresponding eigenvectors are $s_1, \dots s_n$.

Since A is a symmetric matrix, it has n (orthonormal) eigenvectors, so we can use the following theorem from linear algebra (the proof is not presented here, but can be found in several Linear Algebra textbooks).

Theorem 2.3 (Theorem of Sturm[2])**.** *Let A be an $n \times n$ real matrix with n-many linearly independent eigenvectors. Then if we define a matrix B in a way that $B := (s_1, \dots, s_n)$ (so its columns are the eigenvectors of A) which is an ortogonal matrix ($B^T = B^{-1}$), then*

$$B^T A B = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Let us define the following matrix:

$$D := \begin{pmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{pmatrix},$$

in which

$$d_{jj} = \begin{cases} \dfrac{1}{\sqrt{|\lambda_j|}}, & \text{if } \lambda_j \neq 0 \\ \\ 1, & \text{if } \lambda_j = 0. \end{cases}$$

Then, if we define a new matrix $C := BD$, the following holds:

$$C^T A C = (BD)^T A\, (BD) = D^T B^T A\, BD =$$

now we use the fact that D is a diagonal matrix, so $D^T = D$:

$$= D(B^T A B)D =$$

by Theorem 2.3,

$$= D \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} D$$

[2]This is usually called "the eigendecomposition of matrices", or "főtengelytétel" in Hungarian.

Now let us calculate the (diagonal) elements of this matrix, which are in the form

$$\frac{1}{\sqrt{|\lambda_j|}}\lambda_j\frac{1}{\sqrt{|\lambda_j|}} = \frac{\lambda_j}{\sqrt{\lambda_j^2}} = \frac{\lambda_j}{|\lambda_j|} = \operatorname{sgn}(\lambda_j).$$

Here sgn is the signum (or sign) function, defined as:

$$\operatorname{sgn}(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Finally, the thing we get is:

$$C^TAC = \begin{pmatrix} \operatorname{sgn}(\lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & \operatorname{sgn}(\lambda_2) & 0 & \cdots & 0 \\ 0 & 0 & \operatorname{sgn}(\lambda_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \operatorname{sgn}(\lambda_n) \end{pmatrix}.$$

So our main goal is to somehow introduce the matrix C^TAC into our equation, since it has a very nice form (it is a diagonal matrix, and only have ± 1 and 0 elements in its diagonal).

The main idea here is to define a *new variable* $y := C^Tx$, and a *new unknown function* $v(y) := u(x)$ (also, $v(y) = v(C^Tx)$). This means that

$$y_p = \sum_{l=1}^{n} C_{lp}x_l \qquad \text{for } p = 1, \dots n, \tag{2.5}$$

(since the elements in a row of C^T are the elements in a column in C).

Then we can rewrite the derivatives of u using this new function v:

$$\partial_j u(x) = \sum_{p=1}^{n} \partial_p v(y)\partial_j y_p = \sum_{p=1}^{n} \partial_p v(y) C_{jp}. \tag{2.6}$$

At the first step, we used the chain rule of multivariable functions, namely

$$\frac{df(x(t), y(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx(t)}{dt} + \frac{\partial f}{\partial y}\frac{dy(t)}{dt},$$

and then equation (2.5).

If we apply another partial derivative to equation (2.6), we get the following:

$$\partial_j\partial_k u(x) = \sum_{p,q=1}^{n} \partial_p\partial_q v(y) C_{jp}C_{kq}, \qquad (j = 1, \dots n). \tag{2.7}$$

Now we can substitute (2.7) into the left-hand side of our original equation (2.4):

$$\sum_{j,k=1}^{n} a_{j,k}\ \partial_j\partial_k u = \sum_{j,k=1}^{n} a_{j,k} \sum_{p,q=1}^{n} \partial_p\partial_q v(y) C_{jp}C_{kq} =$$

$$= \sum_{p,q=1}^{n} \partial_p\partial_q v(y) \sum_{j,k=1}^{n} a_{j,k}\ C_{jp}C_{kq} = \sum_{p,q=1}^{n} \partial_p\partial_q v(y) \sum_{j,k=1}^{n} C_{jp}\ a_{j,k}\ C_{kq}$$

Now we have to realize that the term $C_{jp}\ a_{j,k}\ C_{kq}$ is actually the qth element in the pth row in the matrix C^TAC, which was proved to be $\operatorname{sgn}(\lambda_p)$ if $p = q$ and is zero otherwise.

To sum it up, we have transformed (2.4) using the new function v into the following equation (in domain $(C^T)^{-1}(\Omega)$):

$$\sum_{p=1}^{n} (\operatorname{sgn}(\lambda_p))\ \partial_p^2 v = G \circ (\operatorname{id}, v, \partial_1 v, \dots, \partial_n v) \tag{2.8}$$

Then equation (2.8) is called the **canonical form of equation** (2.4). It is easy to see that this form has only ± 1 coefficients on its left side, so it is easier to handle.

Depending on the type of our equation, the transformation described before can result in three different canonical forms:

Corollary 2.4. *The canonical form of equation* (2.4) *can have the three following forms:*

1. *If the equation is* ***elliptic*** *(here we mean that $\lambda_i > 0$ for all i: although the equations with all negative eigenvalues are also called elliptic, we can multiply those with -1 and then get this case), then the canonical form is:*

$$\sum_{p=1}^{n} \partial_p^2 v = G \circ (\mathrm{id}, v, \partial_1 v, \dots \partial_n v)\,. \tag{2.9}$$

   Note that the left-hand side of (2.9) *is the same as the left-hand side of the Poisson equation.*

2. *If the equation is* ***hyperbolic*** *(here let us assume that $\lambda_1 > 0$ and all of the other eigenvalues are negative - in the other case we can multiply the equation with -1) the canonical form is*

$$\partial_1^2 v - \sum_{p=2}^{n} \partial_p^2 v = G \circ (\mathrm{id}, v, \partial_1 v, \dots \partial_n v)\,. \tag{2.10}$$

   Note that the left-hand side of (2.10) *is the same as the left-hand side of the wave equation with $c = 1$ (here the first variable is time).*

3. *If the equation is* ***parabolic*** *(here let us assume that $\lambda_1 = 0$ and all of the other eigenvalues are positive - in the other case we can multiply the equation with -1) the canonical form is*

$$\sum_{p=2}^{n} \partial_p^2 v = G \circ (\mathrm{id}, v, \partial_1 v, \dots \partial_n v)\,. \tag{2.11}$$

   Note that the left-hand side of (2.11) *is the same as the left-hand side of the stationary (when $\partial_t v = 0$) heat equation.*

2.3.2 Linear equation with constant coefficients

Let us assume that our equation is in the form

$$\sum_{j,k=1}^{n} a_{j,k}\, \partial_j \partial_k u + \sum_{j=1} b_j \partial_j u + cu = f, \tag{2.12}$$

in which $a_{j,k}, b_j, c \in \mathbb{R}$ are constants and $f : \Omega \to \mathbb{R}$ is a given function (source term).

Similarly as in the semilinear case discussed in the previous section, the high order (i.e. second order) terms can be transformed with the choice $v(y) = u(x)$, and then we get

$$\sum_{j,k=1}^{n} (\mathrm{sgn}(\lambda_p))\, \partial_p^2 v + \sum_{p=1} \beta_p \partial_p v + \gamma u = F. \tag{2.13}$$

Note that the remaining terms we have to deal with are the constants β_p and γ: our goal here is to make most of them disappear.

The key idea here is the introduction of the function

$$V(y) := v(y) \exp\left(\sum_{l=1}^{n} \alpha_l y_l \right), \tag{2.14}$$

in which the parameters α_l will be chosen in a way that most of the terms in (2.13) will disappear. Note that we can easily get $v(y)$ from (2.14) if we multiply both sides by $\exp\left(-\sum_{l=1}^{n} \alpha_l y_l\right)$.

By using the chain rule on the first order derivative in (2.13), we get

$$\partial_p v(y) = [\partial_p V(y) - \alpha_p V(y)] \exp\left(- \sum_{l=1}^{n} \alpha_l y_l \right).$$

The second order terms can be calculated similarly:

$$\partial_p^2 v(y) = \left[\partial_p^2 V(y) - 2\,\alpha_p\,\partial_p V(y) + \alpha_p^2\,V(y)\right]\,\exp\left(-\sum_{l=1}^{n}\alpha_l y_l\right).$$

If we substitute these into (2.13), we get

$$\begin{aligned}\sum_{j,k=1}^{n}(\mathrm{sgn}\lambda_p)\,\partial_p^2 v + \sum_{p=1}[\beta_p - 2\alpha_p(\mathrm{sgn}\lambda_p)]\,\partial_p V(y) + \\ + \left[\gamma - \sum_{p=1}^{n}\beta_p\alpha_p + \sum_{p=1}^{n}(\mathrm{sgn}\lambda_p)\,\alpha_p^2\right]V(y) = F\exp\left(\sum_{l=1}^{n}\alpha_l y_l\right).\end{aligned} \tag{2.15}$$

Now let us use the following choices for α_p:

- If $\lambda_p \neq 0$, then let $\alpha_p = \dfrac{\beta_p}{2\,\mathrm{sgn}(\lambda_p)}$: in this way the terms $\partial_p V$ disappear when $\lambda_p \neq 0$.
- If $\lambda_p = 0$ but $\beta_p \neq 0$, then the term V can be made disappear.

Then, by appropriate choice of the parameters α_p, equation (2.12) can be transformed to the following form when all the eigenvalues are non-zero:

$$\sum_{j,k=1}^{n}(\mathrm{sgn}(\lambda_p))\,\partial_p^2 V + dV = G, \tag{2.16}$$

(here d is a new constant, and G is a new, given function). When there are some eigenvalues which are zero, the equation has the form

$$\sum_{j,k=1}^{n}(\mathrm{sgn}\lambda_p)\,\partial_p^2 V + \sum_{p,\text{ if }\lambda_p=0}\beta_p\partial_p V = G. \tag{2.17}$$

We say that equations (2.16) and (2.17) are **the canonical forms of** (2.12).

So the process of choosing the parameters α_p is as follows:

1. First let us calculate those α_p values which correspond to non-zero eigenvalues. Then the coefficient of the $\partial_p V$ is zero.
2. Then determine the remaining ones in a way that the coefficient of V is zero.

Depending on the type of our equation, the transformation described before can result in three different canonical forms:

Corollary 2.5. *The canonical form of equation* (2.12) *can have the three following forms:*

1. *If the equation is* ***elliptic*** *(here we mean that $\lambda_i > 0$ for all i: although the equations with all negative eigenvalues are also called elliptic, we can multiply those by -1 and then get this case), then the canonical form is:*

$$\sum_{p=1}^{n}\partial_p^2 V + dV = G, \tag{2.18}$$

or, in short form:

$$\Delta V + dV = G. \tag{2.19}$$

If $G = 0$, then this equation is called the ***Helmholtz equation****.*

2. *If the equation is* ***hyperbolic*** *(here let us assume that $\lambda_1 > 0$ and all of the other eigenvalues are negative - in the other case we can multiply the equation with -1) the canonical form is*

$$\partial_1^2 V - \sum_{p=2}^{n}\partial_p^2 V + dV = G, \tag{2.20}$$

or, in short form,

$$\partial_1^2 V - \Delta V + dV = G, \tag{2.21}$$

(Note that the Laplace operator is not taken in the first variable.)

3. *If the equation is* ***parabolic*** *(here let us assume that* $\lambda_1 = 0$ *and all of the other eigenvalues are negative - in the other case we can multiply the equation with* -1*) the canonical form is*

$$\beta_1 \partial_1 V - \sum_{p=2}^{n} \partial_p^2 V = G. \tag{2.22}$$

If $\beta_1 = 0$*, then we get an elliptic equation. When* $\beta_1 \neq 0$*, then we can transform the equation again, and in the end we get the following form:*

$$\partial_1 \tilde{V} - \sum_{p=2}^{n} \partial_p^2 \tilde{V} = \tilde{G}. \tag{2.23}$$

or, in short form,

$$\partial_1 \tilde{V} - \Delta \tilde{V} = \tilde{G}. \tag{2.24}$$

Note that equation (2.24) *is the same as the heat equation, meaning that we could transform a general parabolic equation to the heat equation, so it is enough to examine the latter one, since the general properties of the equation do not change during the transformation.*

In the next chapter we define a useful set of functions, which will be used extensively in the following chapters involving distributions.

Chapter 3

The set $C_0^\infty(\Omega)$

In this chapter we define a useful set of functions, which will be widely used in Chapter 4, and then prove some useful results about their properties.

3.1 Introduction

First we refresh the notation of multiindex, and then define the set of continuously differentiable functions.

3.1.1 Notation

As it was defined in Section 1.1.1, a **multiindex** is a vector in the form $(\alpha_1, \alpha_2, \dots \alpha_N)$ which collects all the orders of different partial derivatives, namely

$$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_N} f,$$

in which $f : \mathbb{R}^N \to \mathbb{R}$ is a (sufficiently smooth) function. We have also defined the "absolute value" of multiindices in the following way:

$$|\alpha| = \alpha_1 + \dots + \alpha_N.$$

We can also define the **sum of multiindices**: if $\alpha = (\alpha_1, \alpha_2, \dots \alpha_N)$ and $\beta = (\beta_1, \dots, \beta_N)$, then

$$\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_N + \beta_N).$$

Also, an **ordering of multiindices** can also be defined: we say that $\alpha \geq \beta$, if $\alpha_j \geq \beta_j$ for every j index. Then, if $\alpha \geq \beta$, then we can also compute the **difference** of two such multiindices:

$$\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_N - \beta_N).$$

Then, the last definition in this section is the **factorial** of multiindices:

$$\alpha! = (\alpha_1!, \alpha_2!, \dots, \alpha_N!)$$

3.1.2 The space of smooth functions

In this section we suppose that $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an arbitrary, non-empty open set.

Definition 3.1. Let us denote by $C^k(\Omega)$ the set of k-times $(0 \leq k \leq \infty)$ continuously differentiable, real-valued functions defined on Ω.

It is a well known fact that the set $C^k(\Omega)$ is a vector space with the usual function addition and product taken with a scalar.

If $k = \infty$, then we can define the set $C^\infty(\Omega)$ which consist of all of those $\Omega \to \mathbb{R}$ functions which can be continuously differentiated any number of times. In other words,

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega).$$

When $k = 0$, then we get the set of continuous functions $C(\Omega)$ (and we will use this notation instead of $C^0(\Omega)$).

We can also define the previous sets on closed domains:

Definition 3.2. Let $C(\overline{\Omega})$ be the set of continuous $\overline{\Omega} \to \mathbb{R}$ functions.

Then $C^k(\overline{\Omega})$ is the vector space of those $f : \overline{\Omega} \to \mathbb{R}$ functions, for which:

- f is in $C^k(\Omega)$ (so they are k times continuously differentiable in the interior of the set), and
- for every $|\alpha| \leq k$ multiindex $\partial^\alpha f \in C(\overline{\Omega})$ in the sense that there exists a continuous extension[1] of this function onto the set $\overline{\Omega}$.

The next one is a useful tool used in the calculation of partial derivatives.

Theorem 3.1 (Leibniz rule)**.** *Let $f, g \in C^k(\Omega)$ functions and α multiindex for which $|\alpha| \leq k$. Then*

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(\partial^\beta f\right) \left(\partial^{\alpha-\beta} g\right),$$

in which $\binom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$ (the division and multiplication is meant to be computed element-wise).

3.2 The set of smooth functions with compact support

Before our main definition, we have to define what a support of a function is:

Definition 3.3. The **support**[2] of an $f \in C(\Omega)$ function is defined as:

$$\operatorname{supp}(f) := \Omega \setminus \{x \in \Omega : \exists\, U_x \subset \Omega \text{ neighborhood of } x, \text{ in which } f = 0 \text{ on } U_x\}.$$

So the support is the collection of those points which *do not* have a neighborhood around them in which the function is zero. This usually means that the support is those points at which the function is not zero, and the closure of these sets.

(Note that the support of measurable function can be defined too, in this case we need "$f = 0$ almost everywhere on U_x" in the definition.)

Definition 3.4. For $0 \leq k \leq \infty$, let $C_0^k(\Omega)$ be the vector space of those $f \in C^k(\Omega)$ functions, for which $\operatorname{supp}(f)$ is compact in $\mathbb{R}^n$.

In distribution theory, the case $k = \infty$ is widely used, as we will see in Chapter 4. A natural question is whether such $f \in C_0^\infty(\Omega)$ functions exist or not. The next proposition answers this question, and also shows an important set of such functions.

[1] Here *continuous extension* means that the values of the function at the boundary points are defined as the limit of the function.

[2] "Tartó" in Hungarian.

Proposition 3.2. *Let $a \in \mathbb{R}^n$, $r > 0$ and we define the function $\eta_{a,r} : \mathbb{R}^n \to \mathbb{R}$ as:*

$$\eta_{a,r}(x) := \begin{cases} \exp\left(\dfrac{-1}{r^2 - |x-a|^2}\right), & \text{if } |x-a| < r, \\ 0, & \text{if } |x-a| \geq r. \end{cases}$$

Then $\eta_{a,r} \in C_0^\infty(\mathbb{R}^n)$.

Proof. (Only main ideas)

Let us define the function

$$h(t) := \begin{cases} \exp\left(-\dfrac{1}{t}\right), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

and $g(x) := r^2 - |x-a|^2$ ($x \in \mathbb{R}^n$, and a and r are parameters). Then $\eta_{a,r} = h \circ g$.

It can be shown that $g \in C^\infty(\mathbb{R}^n)$, and also $h \in C^\infty(\mathbb{R}^n)$, which means that $\eta_{a,r} = h \circ g \in C^\infty(\mathbb{R}^n)$.

Also, $\text{supp}(\eta_{a,r}) = \overline{B(a,r)}$ (the r-radius open ball with center at a), meaning that $\eta_{a,r} \in C_0^\infty(\mathbb{R}^n)$. □

The plot of function $\eta_{a,r}$ can be seen on Figure 3.1.

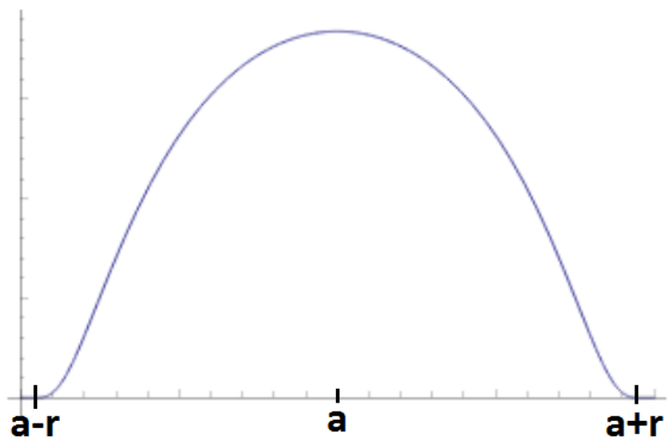


Figure 3.1: Function $\eta_{a,r}$. The maximum of the function is at point $x = a$, and it is nonzero on the interval $(a - r, a + r)$.

Remark 3.1. An interesting property of the function defined above is that it is non-negative.

Now we define a useful set of functions which will be extensively used in the theory of distributions.

Definition 3.5. Let us suppose that for a set of functions f_ε (indexed by the parameter $\varepsilon \in I \subset \mathbb{R}^+$), the following conditions hold:

1. $f_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ for every value of ε,
2. $f_\varepsilon \geq 0$ for every value of ε,
3. $\operatorname{supp}(f_\varepsilon) = \overline{B(0,\varepsilon)}$ for every value of ε,
4. $\displaystyle\int_{\mathbb{R}^n} f_\varepsilon = 1$ for every value of ε.

Then this set of functions is called a **set of mollifiers**[3].

Remark 3.2. The name "mollifiers" comes from the mathematician Kurt Otto Friedrichs (1901-1982), who named them in 1944 after his colleague Donald Alexander Flanders (1900-1958), who had the nickname Moll after the novel Moll Flanders written by Daniel Defoe. Also, the verb "mollify" means to pacify, relieve, temper or lessen, which is also usually the effect of these functions when some other function is multiplied by them. However, such functions were also used before by Sobolev in his 1938 paper, so it was not Friedrichs who first defined them, but the first who named them.

Remark 3.3. Let us consider the functions $\eta_{a,r}$ defined in Proposition 3.2 with $a = 0$ and $r = 1$. If we use the transformation

$$\eta_\varepsilon(x) := \eta_{0,1}\left(\frac{x}{\varepsilon}\right)\frac{C}{\varepsilon^n}$$

in which C is a constant chosen in a way that $\displaystyle\int_{\mathbb{R}^n} \eta_\varepsilon = 1$, then η_ε is a set of mollifiers.

Remark 3.4. Sometimes the elements in a set of mollifiers are called the **approximations of the identity**. The reason for this name is the following.

Let us examine the behavior of the mollifiers when $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0}\eta_\varepsilon(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases}$$

However, we know that the integral of all of the mollifiers is one, meaning that the "limit" of the mollifiers is a weird mathematical object: it is zero everywhere except at zero where it is infinity, and it has an integral of one. If the reader is familiar with the physical concept of the point mass, this would the "density function" of such object. Usually this object is called the **Dirac delta**, which is of course not a function in the classical sense, but in Chapter 4 we will see that it can be defined in a way that it can be well-defined in the mathematical sense too. It will be discussed it Section 4.6 that in the algebraic structure of the set of distributions with the convolution operation, the Dirac delta mentioned above is an identity element, so since mollifiers "approximate" this mathematical object, they are the approximations of the identity in this sense.

Also, the same object can be thought of as the "density function" of a discrete distribution, hence the name "distribution", which will be given to them in the next chapter.

[3] "Egységapproximációt generáló függvények" in Hungarian.

Finally, we mention a useful result from functional analysis.

Theorem 3.3. *If $1 \le p < \infty$, then $C(\Omega)$ is a dense subset of $L^p(\Omega)$*

Remark 3.5. Note that the theorem above is not true when $p = \infty$, i.e. for L^∞ spaces (the set of bounded functions): one can easily construct a function which belongs to L^∞, e.g.

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \ge 0. \end{cases}$$

($f \in L^\infty$ since it is bounded), but we cannot approximate f with continuous functions.

In the next sections we present three different applications of mollifiers, which will be useful in the next chapter.

3.3 Applications of mollifiers I: C_0^∞ is dense in L^p

The aim of this section is to prove the following useful result.

Theorem 3.4 (Approximation theorem)**.** *Let $f \in L^1(\Omega)$, and $\varepsilon > 0$ be an arbitrary value.*

Let $f_\varepsilon : \Omega \to \mathbb{R}$ be the following function:

$$f_\varepsilon(x) := \int_\Omega f(y)\eta_\varepsilon(x-y)dy \qquad (x \in \Omega),$$

in which η_ε are a set of mollifiers. (This is actually the convolution $f * \eta_\varepsilon$, see Section 4.6.1.)

Then, the following statements are true:

a) For every value of $\varepsilon > 0$, f_ε is well-defined (the integral exists), $f_\varepsilon \in C^\infty(\Omega)$, and if $supp(f) \subset \Omega$ is a compact set, then $f_\varepsilon \in C_0^\infty(\Omega)$ for small values of ε.

b) If $\varepsilon \to 0+$, then $f_\varepsilon \to f$ almost everywhere on Ω.

c) If $f \in C(\Omega)$, then $f_\varepsilon \to f$ uniformly on all compact subsets of Ω.

d) If $f \in L^p_{loc}(\Omega)$ $(1 \le p < \infty)$, then for all $K \subset \Omega$ compact sets $f_\varepsilon \to f$ if $\varepsilon \to 0+$ in the norm of $L^p(K)$.

Proof. We prove the four statements separately.

a) Since $|f(y)\eta_\varepsilon(x-y)| \le \text{const}|f(y)|$ (since η_ε is bounded) and $f \in L^1(\Omega)$, the integral in the definition of f_ε exists.

Since the function $y \to \eta_\varepsilon(x-y)$ is in $C^\infty(\Omega)$ and $f \in L^1(\Omega)$, then from the theory of parametric integrals we get that $f_\varepsilon \in C^\infty$.

For the last part (f_ε has compact support), assume that $\text{supp}(f) = K$ is a compact set in Ω, and let $\varepsilon < \text{dist}(K, \partial\Omega)$ be an arbitrary value, and

$$K_\varepsilon := \{x \in \mathbb{R}^n : \text{dist}(x, K) \le \varepsilon\}.$$

(So K_ε is a neighborhood around K, with a boundary not further from the boundary of K than ε.)

Our goal now is to show that if $x \in \Omega \setminus K_\varepsilon$, then $y \to f(y)\eta_\varepsilon(x-y) = 0$. If we can prove this, then we are finished with this part, since then $f_\varepsilon = 0$ at those points, meaning that $f_\varepsilon \in C_0^\infty(\Omega)$.

We know that $f(y)\eta_\varepsilon(x-y) \ne 0$ only if $y \in \text{supp}(f) = K$ and $y \in \text{supp}(z \to \varepsilon(x-z))$ holds at the same time - the last one means that $y \in \overline{B(x,\varepsilon)}$. However, if $x \in \Omega \setminus K_\varepsilon$, then $K \cap \overline{B(x,\varepsilon)} = \emptyset$ (since x is outside of K_ε, meaning that its distance is at least ε from K), but that means that $f(y)\eta_\varepsilon(x-y) = 0$, so the statement is proved.

b) In this part we use the theorem of Lebesgue points:

Theorem 3.5 (Theorem of Lebesgue points)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$, we get that*

$$\lim_{r\to 0+} \frac{1}{\mathrm{vol}(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|dy = 0.$$

(Here the term $\mathrm{vol}(B(x,r))$ is sometimes replaced by r^2.)

(The proof is not presented here.)

Let $x \in \Omega$ be a fixed point. Then since $\int_{\mathbb{R}^n} \eta_\varepsilon = 1$, we get that

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x-y)dy = 1$$

and by multiplying both sides by $f(x)$,

$$f(x) = f(x)\int_{\mathbb{R}^n} \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^n} f(x)\eta_\varepsilon(x-y)dy = \int_{B(x,\varepsilon)} f(x)\eta_\varepsilon(x-y)dy$$

in the last step we used that the integral is zero outside of the support of the function.

Then if we consider the difference between f and f_ε:

$$|f_\varepsilon(x) - f(x)| = \left| \int_{B(x,\varepsilon)} (f(y) - f(x))\, \eta_\varepsilon(x-y)dy \right| \leq$$

$$\leq \frac{C}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(y) - f(x)|\, \eta_{\,0,1}\left(\frac{x-y}{\varepsilon}\right) dy \leq$$

in which we used the definition of η_ε (see Remark 3.3.) Now we use the fact that the function $\eta_{\,0,1}$ is bounded from above:

$$\leq \frac{\tilde{C}}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(y) - f(x)|dy \to 0, \qquad \text{as } \varepsilon \to 0,$$

in which we used the statement of the theorem of Lebesgue points. Thus, f_ε converges to f almost everywhere as $\varepsilon \to 0$.

c) Assume that $f \in C(\Omega)$ and $K \subset U$ is a compact set. Then by the proof of part b):

$$|f_\varepsilon(x) - f(x)| = \left| \int_{B(x,\varepsilon)} (f(y) - f(x))\, \eta_\varepsilon(x-y)dy \right| \leq$$

$$\leq \int_{B(x,\varepsilon)} |f(y) - f(x)|\, \eta_\varepsilon(x-y)dy$$

Since $f \in C(\Omega)$, then by the theorem of Heine, f is uniformly continuous on K. This means that for all $\nu > 0$ values there exists a $\delta > 0$ value such that if $|x-y| < \delta$ (for $x, y \in K$), then $|f(x) - f(y)| < \nu$.

So if we fix ν, then with the choice of $\delta = 2\varepsilon$,

$$|f_\varepsilon(x) - f(x)| = \int_{B(x,\varepsilon)} |f(y) - f(x)|\, \eta_\varepsilon(x-y)dy \leq$$

$$\leq \nu \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)dy = \nu,$$

where we used that η_ε is a mollifier. By definition, this means that f_ε converges uniformly to f.

d) We prove this part in six steps.

1. **Step 1:** Let us assume that $f \in L^p_{loc}(\Omega)$ $(1 \leq p < \infty)$, and let $V \subset \Omega$ be an arbitrary bounded open set for which $\overline{V} \subset \Omega$. Now we state the following theorem coming from set theory.

Proposition 3.6. *Let U be open, and V be an open and bounded set in a way that $\overline{V} \subset U$. Then there is an open and bounded set W for which*

$$\overline{V} \subset W \subset \overline{W} \subset U.$$

Proof. (Proposition 3.6)

Let $d := \operatorname{dist}(\overline{V}, \partial U)$ (here ∂U is the boundary of U). Now $d > 0$ since $\overline{V}$ is closed, and ∂U is bounded since it is closed and bounded.

Let us consider the set $W := \overline{V}_{d/2}$, which is the open neighborhood of V with a boundary which is at the distance $d/2$ from the boundary of V. Then the set W fulfills the statement of the proposition, since W is bounded (since V is bounded), it is open, and $V \subset W$. Also,

$$\overline{W} = \left\{x \in \mathbb{R} : \operatorname{dist}(x, \overline{V}) \leq \operatorname{dist}(\overline{V}, U)/2\right\} \subset U,$$

which proves the statement. □

(Continuation of the proof of Theorem 3.4.)

Now we show it in Steps 2 and 3 that

$$\|f_\varepsilon\|_{L^p(V)} \leq \|f\|_{L^p(W)}. \tag{3.1}$$

2. **Step 2:** This is the proof of the inequality (3.1) for $p = 1$.

$$\|f_\varepsilon\|_{L^1(V)} = \left| \int_V \int_\Omega f(y)\eta_\varepsilon(x-y)dydx \right| \leq$$

By the theorem of Fubini:

$$\leq \int_V \int_\Omega |f(y)|\eta_\varepsilon(x-y)dydx = \int_V |f(y)| \left(\int_\Omega \eta_\varepsilon(x-y)dx \right) dy =$$

$$= \int_V |f(y)|dy = \|f\|_{L^1(V)} \leq \|f\|_{L^1(W)}.$$

3. **Step 3:** This is the proof of the inequality (3.1) for $p > 1$.

$$|f_\varepsilon| = \left| \int_\Omega f(y)\eta_\varepsilon(x-y)dy \right| \leq$$

By the theorem of Fubini:

$$\leq \int_\Omega |f(y)|\eta_\varepsilon(x-y)dy = \int_\Omega |f(y)|\eta_\varepsilon^{1/p}(x-y)\eta_\varepsilon^{1/q}(x-y)dy \leq$$

By the inequality of Hölder:

$$\leq \left(\int_\Omega |f(y)|^p \eta_\varepsilon(x-y)dy \right)^{1/p} \left(\int_\Omega \eta_\varepsilon(x-y)dy \right)^{1/q}.$$

in which $1/p + 1/q = 1$.

Then by definition:

$$\|f_\varepsilon\|_{L^p(V)}^p = \int_V |f_\varepsilon(x)|^p dx \leq \int_V \left(\int_\Omega |f(y)|^p \eta_\varepsilon(x-y)dy \right) dx \leq$$

in which we used the previous bound and the fact that $\int_\Omega \eta_\varepsilon(x-y)dy = 1$. By Fubini's theorem:

$$= \int_V |f(y)|^p \int_\Omega \eta_\varepsilon(x-y)dydx =$$

$$= \int_V |f(y)|^p dy \leq \int_W |f(y)|^p dy,$$

from which we get (3.1).

4. **Step 4:** By Theorem 3.3 we know that $C(\Omega)$ is dense in $L^p(\Omega)$, meaning that for all values of $\mu > 0$ there is such a function $g \in C(W)$ for which

$$\|f - g\|_{L^p(W)} < \nu. \tag{3.2}$$

5. **Step 5:** Let us define functions g_ε similarly as f_ε. From part b) we know that $g_\varepsilon \to g$ uniformly on V, and we have also proved that $\|g_\varepsilon\|_{L^p(V)} \leq \|g\|_{L^p(W)}$, so functions g_ε can be bounded form above, meaning that by the theorem of Lebesgue $g_\varepsilon \to g$ holds also in the norm of $L^p(W)$.

6. **Step 6:** Now we prove the statement of part d).

$$\|f_\varepsilon - f\|_{L^p(V)} = \|(f_\varepsilon - g_\varepsilon) + (g_\varepsilon - g) + (g - f)\|_{L^p(V)} \leq$$
$$\leq \|f_\varepsilon - g_\varepsilon\|_{L^p(V)} + \|g_\varepsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \tag{3.3}$$

   Now we bound these terms separately.

   - *First term:* By (3.1):

$$\|f_\varepsilon - g_\varepsilon\|_{L^p(V)} = \|(f - g)_\varepsilon\|_{L^p(V)} \leq \|f - g\|_{L^p(W)} \leq \nu$$

   - *Second term:* Since $g_\varepsilon \to g$ holds uniformly,

$$\|g_\varepsilon - g\|_{L^p(V)} \leq \nu$$

     when ε is small enough.
   - *Third term*: By (3.2) and $V \subset W$,

$$\|g - f\|_{L^p(V)} \leq \|g - f\|_{L^p(W)} \leq \nu.$$

   So this means that the terms in (3.3) are smaller than 3ν when ε is small enough. Since ν can be arbitrary small, then $f_\varepsilon \to f$ in the norm of $L^p(V)$, but it is the same as the norm of $L^p(\overline{V})$, in which $\overline{V}$ is an arbitrary compact set, so our statement is proved.

□

The following theorem can be proved using the previous result:

Theorem 3.7. *If $1 \leq p < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$*

Proof. Let $f \in L^p(\Omega)$ be a given function. We prove the statement in four parts.

1. **Step 1:** Let us consider the sets

$$\Omega^\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\} \cap B(0,\ 1/\delta),$$

   which are non-empty for a sufficiently small δ value (by the proof of Theorem 3.4). Let χ_δ be the characteristic function of Ω_δ, meaning that $\chi_\delta(x) = 1$ if $x \in \Omega_\delta$ and 0 otherwise.

2. **Step 2:** $f\chi_\delta \to f$ a.e. on the set Ω as $\delta \to 0+$. Also, $\|f\chi_\delta\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}$, so by the theorem of Lebesgue $f\chi_\delta \to f$ also in the norm of $L^p(\Omega)$. Then we can choose a small number $\delta > 0$ for which $\|f\chi_\delta - f\|_{L^p(\Omega)} < \nu$ holds for some given $\nu > 0$ value.

3. **Step 3:** Since $g := f\chi_\delta$ is a function with compact support (its support is part of the ball $B(0,\ 1/\delta)$), so by Theorem 3.4 for functions g_ε we have $g_\varepsilon \in C_0^\infty(\Omega)$ and $g_\varepsilon \to g$ as $\varepsilon \to 0+$ in the norm of $L^p(\Omega)$.

4. **Step 4:** For a sufficiently small ε value we have $\|g_\varepsilon - g\|_{L^p(\Omega)} < \nu$, and then

$$\|f - g_\varepsilon\|_{L^p(\Omega)} \leq \|f - g\|_{L^p(\Omega)} + \|g - g_\varepsilon\|_{L^p(\Omega)} < 2\nu,$$

   in which $\nu > 0$ was arbitrary, meaning that f can be approximated by functions from $C_0^\infty(\Omega)$, which proves our theorem.

□

3.4 Application of mollifiers II: construction of some special functions

In this section we prove a result which will be used in the proof of the theorem of Section 3.5.

Theorem 3.8. *Let $K \subset \Omega$ be compact.*

Then, there exists a $\varphi \in C_0^\infty(\Omega)$ function such that $0 \leq \varphi \leq 1$, and $\varphi = 1$ in a neighborhood of K.

Proof. Let $d := \operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$. This value is positive (not zero), since $K \subset \Omega = \operatorname{Int}(\Omega)$.

Let us fix a value $0 < \varepsilon < \dfrac{d}{3}$ and the set $\overline{K_{3\varepsilon}} := \{x \in \Omega : \operatorname{dist}(x, K) \leq 3\varepsilon\}$ (so a neighborhood of K which has a boundary at an 3ε distance from K).

Then, $\overline{K_{3\varepsilon}} \subset \Omega$ (by the definition of ε), and let us define the function

$$f(x) := \begin{cases} 1, & \text{if } x \in \overline{K_{2\varepsilon}}, \\ 0, & \text{if } x \in \Omega \setminus K_{2\varepsilon}. \end{cases}$$

Now we show that the function f_ε (the modification of the function f defined above in the way described in the Approximation Theorem (Theorem 3.4)) is a proper choice for φ (the function of the statement).

By the Approximation Theorem, $f_\varepsilon \in C_0^\infty(\Omega)$. Moreover, if $x \in \overline{K_\varepsilon}$, then $B(x, \varepsilon) \subset \overline{K_{2\varepsilon}}$, meaning that

$$f_\varepsilon(x) = \int_{B(x,\varepsilon)} f(y)\eta_\varepsilon(x - y)dy =$$

Now we use the definition of function f:

$$= \int_{B(x,\varepsilon)} 1 \cdot \eta_\varepsilon(x - y)dy = 1,$$

by the definition of η_ε. This means that $f_\varepsilon = 1$ on $\overline{K_\varepsilon}$ (a neighborhood of K).

Moreover, since $0 \leq f \leq 1$ and $\eta_\varepsilon \geq 0$,

$$0 \leq f_\varepsilon(x) \leq \int_{\mathbb{R}^n} \eta_\varepsilon(x - y)dy = 1.$$

Then, $\varphi := f_\varepsilon$ is a proper choice. □

3.5 Smooth partition of unity

In this section we state a useful theorem which will be used later in one of our proofs.

Theorem 3.9 (Smooth partition of unity[4])**.** *Let $K \subset \mathbb{R}^n$ be a compact set, and $\Omega_j \subset \mathbb{R}^n$ open sets in a way that $K \subset \bigcup_{j=1}^m \Omega_j$ (such sets exist since K is compact).*

Then there exists a set of $\varphi_j \in C_0^\infty(\Omega_j)$ $(j = 1, \ldots, n)$ functions, for which $\sum_{j=1}^m \varphi_j = 1$ in a neighborhood of K.

[4]"Egységosztás tétele" in Hungarian.

Proof. (of Theorem 3.9) The proof consists of three parts: first, we construct a cover of K in which all of the sets are bounded. Then we apply Theorem 3.8 on the closure of these sets which results in the existence of a set of functions ψ_j, and then construct the functions φ_j from the functions ψ_j.

1. **Step 1:** Our goal in this part is to prove the following lemma:

   Lemma 3.10. *There exists a collection of bounded open sets G_j in a way that $\overline{G_j} \subset \Omega_j$ and $K \subset \bigcup_{j=1}^m G_j$.*

   Proof. (of Lemma 3.10)

   For the proof, we are using the following result of set theory.

   Proposition 3.11. *Let U be open, and V be an open and bounded set in a way that $\overline{V} \subset U$. Then there is an open and bounded set W for which*

   $$\overline{V} \subset W \subset \overline{W} \subset U.$$

   (The proof of this result is not presented here, but can be found in the proof of part d) of Theorem 3.4. It is not part of the exam materials.)

   Then, if we take $K \setminus \bigcup_{j=2}^m \Omega_j$ and Ω_1, then we can apply Proposition 3.11 with $U = \Omega_1$ and $V = K \setminus \bigcup_{j=2}^m \Omega_j$, which means that there exists a set G_1, for which

   $$\overline{K \setminus \bigcup_{j=2}^{m} \Omega_j} \subset G_1 \subset \overline{G_1} \subset \Omega_1.$$

   In the next step, let us make the choice $U = \Omega_2$ and $V = K \setminus \left(\bigcup_{j=3}^m \Omega_j \cup G_1\right)$. Then Proposition 3.11 can applied again (since V is bounded), meaning that there exists a bounded set G_2, for which

   $$\overline{K \setminus \left(\bigcup_{j=3}^{m} \Omega_j \cup G_1\right)} \subset G_2 \subset \overline{G_2} \subset \Omega_2.$$

   Then by induction we get a list of sets $G_1, \dots G_m$, and these fulfill the statement of Lemma 3.10, since $\overline{G_j} \subset \Omega_j$, and $\nexists\ x \in K$ for which $x \notin G_j$ for some j, since the last step in the induction is

   $$\overline{K \setminus \left(\bigcup_{j=1}^{m-1} G_j\right)} \subset G_m \subset \overline{G_m} \subset \Omega_m,$$

   meaning that if $x \in K$ but $x \notin G_m$, then

   $$x \in (K \setminus G_m) \ \subset \ K \setminus \left(\overline{K \setminus \left(\bigcup_{j=1}^{m-1} G_j\right)}\right) = K \cap \left(\bigcup_{j=1}^{m-1} G_j\right)$$

   Then this means that $x \in \left(\bigcup_{j=1}^{m-1} G_j\right)$, so there is an index j for which $x \in G_j$. □

   So we have proved Lemma 3.10.

2. **Step 2:** We apply Theorem 3.8 to the sets $\overline{G_j} \subset \Omega_j$. This can be done, since the sets G_j are compact: the reason for this is that since we are in $\mathbb{R}^n$, the bounded and closed properties imply compactness.

   Then, by Theorem 3.8 we know that there exist some functions $\psi_j \in C_0^\infty(\Omega_j)$ in a way that $\psi_j = 1$ in a neighborhood of $\overline{G_j}$.

3. **Step 3:** In this step we construct the functions φ_j.

   Let us define the following functions:

$$\begin{aligned}
\varphi_1 &:= \psi_1, \\
\varphi_2 &:= \psi_2(1-\psi_1), \\
\varphi_3 &:= \psi_3(1-\psi_2)(1-\psi_1), \\
&\vdots \\
\varphi_m &:= \psi_m(1-\psi_1)(1-\psi_2)\dots(1-\psi_{m-1}).
\end{aligned}$$

   Now we show that these fulfill the statement of the theorem.

   It is easy to see that $\varphi_j \in C_0^\infty(\Omega_j)$ for every j index, and if we substitute the identity

$$\psi_j = 1 - (1-\psi_j)$$

   into the definition of φ_j, we get:

$$\varphi_j = (1-(1-\psi_j))(1-\psi_1)(1-\psi_2)\dots(1-\psi_{j-1}),$$

$$\varphi_j = (1-\psi_1)(1-\psi_2)\dots(1-\psi_{j-1}) - (1-\psi_1)(1-\psi_2)\dots(1-\psi_{j-1})(1-\psi_j).$$

   Then, if we add all of the φ_j terms up, we get a telescoping sum in which all of the terms vanish except for the first and the last one:

$$\begin{aligned}
\sum_{j=1}^{m} \varphi_j = \psi_1 + (1-\psi_1) - (1-\psi_1)(1-\psi_2) + \\
+ (1-\psi_1)(1-\psi_2) - \dots - (1-\psi_1)(1-\psi_2)\dots(1-\psi_m) = \\
= 1 - (1-\psi_1)(1-\psi_2)\dots(1-\psi_m)
\end{aligned}$$

   Now we have to show that the second term is zero on $\sum_{j=1}^m G_j$ (and then since $K \subset \sum_{j=1}^m G_j$, then it is also zero on K).

   Let us examine a point $x_0 \in \bigcup_{j=1}^m G_j$ - then it means that $x \in G_k$ for some index k. By Step 2 of this proof we know that $\psi_k(x_0) = 1$ on a neighborhood of $\overline{G_k}$, which means that

$$(1-\psi_1)(1-\psi_2)\dots(1-\psi_k)\dots(1-\psi_m) = 0,$$

   and therefore $\sum_{j=1}^m \varphi_j = 1$.

Consequently, we have proved the statement of our theorem. □

Remark 3.6. Theorem 3.9 is also true in a much more general context, namely in topological spaces.

Chapter 4

Distribution theory

In this chapter we introduce a new mathematical notion called distribution. Note that although these distributions and probability distributions are not the same, they are not that far either: one can construct a distribution from a probability one. Another important remark is that professor Miklós Horváth has a course called "Distribution theory and Green functions"[1], in which the topics of this chapter are discussed in much more detail (but the course is usually given in Hungarian).

The physical problem which was the beginning of the theory of distributions was the problem of units of impulse. Paul Dirac (1902-1984) in 1927 proposed the following "function":

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases} \tag{4.1}$$

and he also stated that for this function

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

holds. One can think of such "function" as the density of a point mass: it has only density at the point at which it is concentrated (in this case this is $x = 0$), and then the integral of the density function should give its mass, which is one.

Of course, the mathematical object (4.1) is not a function in the classical sense: the object ∞ is not a number, so the value of a function cannot be that one - also, this is not integrable, so its integral cannot be 1. One can say that then Dirac was talking nonsense - however, in applications that function usually makes an appearance in the form

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \qquad \forall f \in L^1,$$

in which case the object $\delta(x)$ can be defined.

Another interesting statement Dirac made was that object $\delta(x)$ is the derivative of the Heaviside function:

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases} \tag{4.2}$$

which is, of course, not differentiable in the classical sense, although one can argue that the function needs to move vertically at $x = 0$ to be continuous,

[1] "Disztribúcióelmélet és Green függvények" in Hungarian.

so its derivative is infinity there (which is, of course, does not make sense in the theory of real functions). His famous sentence regarding these problems was "Why should I refuse to eat a nice lunch, if I cannot understand what happens in my stomach?"

Later, the ideas of Dirac were made into a strict mathematical theory by Sergei Sobolev (1908-1989), and also Laurent Swartz (1915-2002), who supposedly came up with the theory of distributions in one night in November 1944.

4.1 Definition of distributions

From now on $\Omega \subset \mathbb{R}^n$ is an open set.

Definition 4.1. Consider the vector space $C_0^\infty(\Omega)$ with the usual addition and the product with scalar.

Let us say that the **convergence** $\boldsymbol{\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi}$ holds, if

(i) there exists a $K \subset \Omega$ compact set such that

$$\operatorname{supp}(\varphi_j) \subset K \qquad \forall j,$$

and

(ii) for every α multiindex,

$$\partial^\alpha \varphi_j \to \partial^\alpha \varphi \qquad \text{uniformly on } \Omega.$$

Then the space $C_0^\infty(\Omega)$ equipped with this convergence is called the **space of test functions**[2] (the reason for this name will be made clear later in Chapter 11), and is denoted by $\mathcal{D}(\Omega)$.

Now we can define distributions.

Definition 4.2. A functional $u : \mathcal{D}(\Omega) \to \mathbb{R}$ is called a **distribution**, if the following hold:

(i) u is linear, and

(ii) u is **sequentially continuous**[3], meaning that if $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$, then $u(\varphi_j) \to u(\varphi)$.

Remark 4.1. In general the continuity of an operator implies the sequential continuous property, but the contrary is only true in a special class of spaces called *sequential spaces*.

The problem with Definition 4.2 is that condition (ii) is hard to check - instead of considering the images of the convergent sequences (in the sense defined above), we can use the following theorem.

[2] "Próbafüggvények tere" in Hungarian.
[3] "Sorozatfolytonos" in Hungarian.

Theorem 4.1. *Let $u : \mathcal{D}(\Omega) \to \mathbb{R}$ be a linear functional. Then the following two properties imply each other.*

(i) u is sequentially continuous.

(ii) For every compact set $K \subset \Omega$ there exists corresponding constants $c_K \in \mathbb{R}^+$ and $m_K \in \mathbb{Z}^+$ such that

$$|u(\varphi)| \leq c_K \sum_{|\alpha| \leq m_K} \sup_K |\partial^\alpha \varphi|$$

for such $\varphi \in \mathcal{D}(\Omega)$ functions for which $\operatorname{supp}(\varphi) \subset K$.

Proof. We prove the two implications separately.

A) The case (ii) $\Rightarrow$ (i).

Suppose that (ii) holds, and our goal is that (i) holds, meaning that if $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$, then $u(\varphi_j) \to u(\varphi)$. So let us assume that (ii) and $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$ holds. $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$ means that there is a compact set $K \subset \Omega$ for which $\operatorname{supp}(\varphi_j) \subset K$ for every index j and for every α multiindex $\partial^\alpha \varphi_j \to \partial^\alpha \varphi$ uniformly on Ω.

Then, if we apply property (ii) on the set K defined above and for the function $\varphi_j - \varphi$, we get:

$$|u(\varphi_j) - u(\varphi)| = |u(\varphi_j - \varphi)| \leq c_K \sum_{|\alpha| \leq m_K} \sup_K |\partial^\alpha \varphi_j - \partial^\alpha \varphi| \tag{4.3}$$

The first equality holds since u is linear. Then the right-hand side of (4.3) tends to zero by the assumption, which means that the left-hand side should also tend to zero, meaning that $u(\varphi_j) \to u(\varphi)$, which finishes this part of the proof.

B) The case (i) $\Rightarrow$ (ii).

Let us suppose that (i) holds, meaning that if $\varphi_j \xrightarrow{\mathcal{D}(\Omega)} \varphi$, then $u(\varphi_j) \to u(\varphi)$. Our goal is to prove that (ii) holds.

We prove this property in the indirect way. Let us assume proceeding towards contradiction that there is such a compact set $K \subset \Omega$ such that for every $C \in \mathbb{R}^+$, $m \in \mathbb{Z}^+$ constants there is such a function $\varphi \in \mathcal{D}(\Omega)$ for which $\operatorname{supp}(\varphi) \subset K$, and

$$|u(\varphi)| > C \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha \varphi|. \tag{4.4}$$

Now we get a contradiction in four steps.

1. **Step 1.** Let us consider a sequence $C = m = j \in \mathbb{Z}^+$. Then if we apply the previous assumption, we get a sequence of functions $\varphi_j \in \mathcal{D}(\Omega)$ for which $\operatorname{supp}(\varphi_j) \subset K$, and inequality (4.4) holds for all of them:

$$|u(\varphi_j)| > j \sum_{|\alpha| \leq j} \sup_K |\partial^\alpha \varphi_j|.$$

2. **Step 2.** Let us define another sequence of functions:

$$\psi_j(x) = \frac{\varphi_j(x)}{j \sum_{|\alpha|\le j} \sup_K |\partial^\alpha \varphi_j|} \qquad \text{for every } x \in \Omega.$$

In Step 3 we show that $u(\psi_j) \to 0$, but in Step 4 we get that $u(\psi_j) > 1$, which will be a contradiction.

3. **Step 3.** It is easy to see that $\psi_j \in \mathcal{D}(K)$, and

$$\sup_K |\partial^\alpha \psi_j| \le \sum_{|\beta|\le j} \sup_K |\partial^\beta \psi_j| = \sum_{|\beta|\le j} \frac{\sup_K |\partial^\beta \varphi_j|}{j \sum_{|\alpha|\le j} \sup_K |\partial^\alpha \varphi_j|} = \frac{1}{j},$$

and if $j \to \infty$, then $\partial^\alpha \psi_j \to 0$ uniformly on K. Since $\operatorname{supp}(\psi_j) \subset K$, then the previous argument means that $\psi_j \xrightarrow{\mathcal{D}(\Omega)} 0$. Then, if property (i) holds, it means that $u(\psi_j) \to 0$.

4. **Step 4.** By the definition of ψ_j and assumption (4.4),

$$u(\psi_j) = \frac{u(\varphi_j)}{j \sum_{|\alpha|\le j} \sup_K |\partial^\alpha \varphi_j|} > 1,$$

so it cannot tend to zero, which gives the contradiction.

Thus, the theorem is proved. □

Since (ii) is easier to check than the definition of sequential continuity, we usually use the former when we prove that a given functional is a distribution.

Remark 4.2. Sometimes there exists a universal constant $m \in \mathbb{Z}^+$ for which

$$|u(\varphi)| \le c_K \sum_{|\alpha|\le m} \sup_K |\partial^\alpha \varphi|$$

on any compact set. Then the smallest one of such m values is called the **order of the distribution**.

Examples of distributions

A) **Regular distributions:** Sometimes these are called generalized functions, and these are the ones which are usually used in the theory of PDEs.

Definition 4.3. Let $f \in L^1_{loc}(\Omega)$. Then the **regular distribution corresponding to function f**, denoted by $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ is defined as

$$T_f(\varphi) := \int_\Omega f\varphi, \qquad \varphi \in \mathcal{D}(\Omega). \tag{4.5}$$

A natural question is whether the functional defined as (4.5) is a distribution. It is easy to see that it is linear (since integration is linear), but for the sequential continuous property we state the following proposition.

Proposition 4.2. *The functional $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ is sequentially continuous.*

Proof. (The same as the solution of Exercise 2. (a) from Practice 4.)

We will use Theorem 4.1, meaning that we have to find an upper bound for the value of the functional using the derivatives of φ. Let us suppose that $\operatorname{supp}(\varphi) \subset K$, and then:

$$|T_f(\varphi)| = \left|\int_\Omega f\varphi\right| = \left|\int_K f\varphi\right| \leq \int_\Omega |f\varphi| \leq \max_K |\varphi| \left(\int_K |f|\right)$$

Since $f \in L^1_{loc}(\Omega)$, we know that $\left(\int_K |f|\right) = c_K < \infty$, so the theorem can be applied. □

A natural question is that if $f \neq g$, then is it possible that $T_f = T_g$? The next proposition answers this one.

Proposition 4.3. *Let $f, g \in L^1_{loc}(\Omega)$ and suppose that $T_f = T_g$, meaning that $T_f(\varphi) = T_g(\varphi)$ for every $\varphi \in \mathcal{D}(\Omega)$. Then $f = g$ almost everywhere.*

Proof. Let $h = f - g$, so $T_h = T_f - T_g = 0$. We prove the statement in 5 steps.

1. **Step 1.** Our goal in this proof is to prove that $h = 0$ almost everywhere on Ω. However, in this step we show that it is enough to prove $h = 0$ on an arbitrary compact set $K \subset \Omega$.

   It is known from set theory that since Ω is an open set, it is a union of countably many compact sets, so if $h = 0$ on all of them, then h might be nonzero on a union of countably many zero measure sets, but this union has also zero measure, so if $h = 0$ almost everywhere on an arbitrary compact set K, then it is also zero almost everywhere on Ω.

2. **Step 2.** Let us define $d = \operatorname{dist}(K, \partial\Omega)$ (in which K is a fixed compact set). We know that $d > 0$, since Ω is open. Then let us define

   $$\tilde{h}(x) := \begin{cases} h(x), & \text{if } x \in K_{d/2}, \\ 0, & \text{if } x \in \Omega \setminus K_{d/2}, \end{cases} \tag{4.6}$$

   in which $\overline{K_{d/2}} := \left\{x \in \Omega : \operatorname{dist}(x, K) \leq \frac{d}{2}\right\}$ (as in the proof of Theorem 3.8). Then $\tilde{h} \in L^1(\Omega)$ and it is zero outside of a compact set (which is $K_{d/2}$).

3. **Step 3.** If we use part (d) of the Approximation Theorem (Theorem 3.4), then we can construct such $\tilde{h}_\varepsilon \in L^1(\Omega)$ functions, for which $\tilde{h}_\varepsilon \to \tilde{h}$ if $\varepsilon \to 0+$ in L^1 norm.

4. **Step 4.** We show in the next step that $\tilde{h}_\varepsilon(x) = 0$ if $x \in K$ and $\varepsilon < d/2$, since if it holds, then when $\varepsilon \to 0$ we get $h(x) = 0$ for all $x \in K$.

5. **Step 5.** Now we prove the claim of Step 4.

   Let $x \in K$. If $\varepsilon < d/2$, then $B(x, \varepsilon) \subset K_{d/2}$, so since the definition of $\tilde{h}_\varepsilon$:

   $$\tilde{h}_\varepsilon(x) = \int_\Omega \tilde{h}(y)\eta_\varepsilon(x - y)dy =$$

   by the definition of $\tilde{h}$:

   $$= \int_{K_{d/2}} h(y)\eta_\varepsilon(x - y)dy =$$

   since $\operatorname{supp}(\eta_\varepsilon) \subset B(0, \varepsilon)$ (so if $y \neq K_{d/2}$, then $\eta_\varepsilon(x - y) = 0$):

   $$= \int_\Omega h(y)\eta_\varepsilon(x - y)dy = T_h(y \to \eta_\varepsilon(x - y)) = 0,$$

   since by assumption $T_h(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$.

This completes the proof. □

Corollary 4.4. *The following two are equivalent:*

(i) $f = g$ *almost everywhere,*

(ii) $T_f = T_g$.

B) **The Dirac-delta function**

Definition 4.4. Let $a \in \mathbb{R}^n$ be a fixed point. Then we define the operator $\delta_a : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}^n$ as

$$\delta_a(\varphi) = \varphi(a).$$

This is called the **Dirac-delta concentrated at point** $\boldsymbol{a}$.

This is also a distribution (see Exercise 3 (a) on Practice 4) but not regular (see Exercise 3 (b) on Practice 4).

4.2 Equivalence and support of distributions

In the previous section we said that two distributions were equal, i.e. $u(\varphi) = v(\varphi)$ if and only if $u(\varphi) = v(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$ test functions. In this section we show another equivalence, and then state a theorem which clears the connection between these two notions. The new notion will be used in the case of the examination of the supports of distributions, which is defined in the second half of this section.

Definition 4.5. Let u and v be two distributions on Ω, and let $G \subset \Omega$ be an open set. Then we say that $u = v$ on G, or they are **globally equivalent** on G, if for all $\varphi \in \mathcal{D}(G)$ test functions $u(\varphi) = v(\varphi)$ holds.

Now we introduce the other notion.

Definition 4.6. Let u and v be two distributions on Ω, and let $G \subset \Omega$ be an open set. Then we say that u and v are **locally equivalent** on G, if for all $x \in G$ points there is a $U_x \subset G$ neighborhood of x, for which $u = v$ globally on U_x.

It is evident that if $u = v$ globally on G, then $u = v$ locally. The next theorem states the interesting fact that the other implication is also true, meaning that these two notions are basically the same.

Theorem 4.5. *Let us consider two distributions* u *and* v *defined on* Ω*, and an open set* $G \subset \Omega$*. Then if* $u = v$ *locally on* G*, then* $u = v$ *globally on* G*.*

Proof. Let us suppose that $G = \Omega$ (the $G \subsetneq \Omega$ case is also similar). Our goal is now to prove that if $u = v$ locally on Ω, then $\forall\ \varphi \in \mathcal{D}(\Omega)$ we have $u(\varphi) = v(\varphi)$.

Let us fix a $\varphi \in \mathcal{D}(\Omega)$ function, and let us use the notation $K := \mathrm{supp}(\varphi)$ (here $K \subset \Omega$ is a compact set).

Since $u = v$ locally, we know that for every point $x \in \Omega$ there is a neighborhood $U_x \subset \Omega$ in which $u = v$ globally. Then $K \subset \bigcup_{x \in \Omega} U_x$, but since K is

compact, every one of its open covers have a finite cover, so there are finitely many x_j points for which

$$K \subset \bigcup_{j=1}^{m} U_{x_j}.$$

Now we can apply the theorem of smooth partition of unity (Theorem 3.9), which means that there are some functions $\varphi_j \in C_0^\infty(\Omega)$ such that $\text{supp}(\varphi_j) \subset U_{x_j}$, and

$$\sum_{j=1}^{m} \varphi_j(x) = 1 \qquad \forall x \text{ in a neighborhood of } K.$$

Then

$$\varphi = \varphi \sum_{j=1}^{m} \varphi_j = \sum_{j=1}^{m} \varphi\varphi_j,$$

and $\text{supp}(\varphi\varphi_j) \subset U_{x_j}$.

Therefore, if we write up the equivalence we would like to prove:

$$u(\varphi) = u\left(\sum_{j=1}^{m} \varphi\varphi_j\right) = \sum_{j=1}^{m} u(\varphi\varphi_j) =$$

where we used that u is linear. Then since $\text{supp}(\varphi\varphi_j) \subset U_{x_j}$, we can apply the local equivalence property:

$$= \sum_{j=1}^{m} v(\varphi\varphi_j) = v\left(\sum_{j=1}^{m} \varphi\varphi_j\right) = v(\varphi),$$

which means that $u = v$ globally. □

An easy consequence of this theorem is that if $u = 0$ locally, then $u = 0$ globally, which is used in the next definition.

Definition 4.7. Let u be a distribution defined on Ω. Then its **support**[4] is defined as

$$\text{supp}(u) := \Omega \backslash \{x \in \Omega : \exists\, U_x \subset \Omega \text{ open neighborhood of } x, \text{ s. t. } u = 0 \text{ on } U_x\}.$$

Note that in the definition the equivalence $u = 0$ is similar to a local one, but by the previous theorem it does not really matter since it is the same as the global equivalence.

Examples:

- $\text{supp}(\delta_a) = \{a\}$.

  Proof. If $a \notin \text{supp}(\varphi)$ (or $a \in \text{supp}(\varphi)$ but $\varphi(a) = 0$), then $\delta_a(\varphi) = \varphi(a) = 0$ (such φ functions are not that interesting).

  In the case of $a \in \text{supp}(\varphi)$ and $\varphi(a) = c \neq 0$, we have $\delta_a(\varphi) = \varphi(a) = c$, meaning that point a does not have a neighborhood around itself where the distribution is zero, so $a \in \text{supp}(\delta_a)$.

[4] "Tartó" in Hungarian

If we consider some other point $b \neq a$, then there is a neighborhood around itself U_b for which $a \notin U_b$. Then $\delta_a(\varphi) = 0$ for all $\varphi \in \mathcal{D}(U_b)$ functions (since $\varphi(a) = 0$ for every $\varphi \in \mathcal{D}(U_b)$ since their support is inside U_b), meaning that $b \notin \operatorname{supp}(\delta_a)$. Consequently, $\operatorname{supp}(\delta_a) = a$. □

- $\operatorname{supp}(T_f) = \operatorname{supp}(f)$.

  Proof. We prove the equality of these sets by showing that they are subsets of each other.

  1. $\operatorname{supp}(T_f) \subset \operatorname{supp}(f)$, or, in other words

     $$x \in \operatorname{supp}(T_f) \;\Rightarrow\; x \in \operatorname{supp}(f),$$

     which is the same as

     $$x \notin \operatorname{supp}(f) \;\Rightarrow\; x \notin \operatorname{supp}(T_f).$$

     If $x \notin \operatorname{supp} f$, then $\exists\ U_x$, such that $f|_{U_x} = 0$, and then $\forall \varphi : \operatorname{supp}(\varphi) \subset U_x$ we have $T_f(\varphi) = \int_{U_x} \varphi f = 0$, meaning that $x \notin \operatorname{supp}(T_f)$.
  2. $\operatorname{supp}(f) \subset \operatorname{supp}(T_f)$, or, in other words

     $$x \in \operatorname{supp}(f) \;\Rightarrow\; x \in \operatorname{supp}(T_f),$$

     which is the same as

     $$x \notin \operatorname{supp}(T_f) \;\Rightarrow\; x \notin \operatorname{supp}(f).$$

     If $x \notin \operatorname{supp}(T_f)$, then $\exists\ U_x$ in a way that in the case of $\operatorname{supp}(\varphi) \subset U_x$ we have $\int_{U_x} \varphi f = 0$ for all $\varphi \in \mathcal{D}(U_x)$, meaning that $T_f = T_0$ on U_x, and therefore $f|_{U_x} = 0$ and then $x \notin \operatorname{supp}(f)$.

  Thus, the statement is proved. □

4.3 Algebraic operations on distributions

In this section we define some algebraic operations on the set of distributions.

Definition 4.8. Let u and v be distributions, and $\lambda \in \mathbb{R}$. Then we define the following operations on them:

$$(u + v)(\varphi) := u(\varphi) + v(\varphi) \qquad (\varphi \in \mathcal{D}(\Omega))$$

$$(\lambda u)(\varphi) := \lambda u(\varphi) \qquad (\varphi \in \mathcal{D}(\Omega))$$

Proposition 4.6. *The set of distributions is a vector space, and this vector space is usually denoted by $\mathcal{D}'(\Omega)$.*

The following is a definition which corresponds to the one used in probability theory.

Definition 4.9. We say that the sequence $u_j \in \mathcal{D}'(\Omega)$ **converges weakly** to $u \in \mathcal{D}'(\Omega)$, if for all $\varphi \in \mathcal{D}(\Omega)$ test functions $u_j(\varphi) \to u(\varphi)$ (as a real sequence). The notation for this convergence is usually $u_j \xrightarrow{\mathcal{D}'(\Omega)} u$.

Note that this notion is similar to the weak convergence defined in other fields of mathematics:

- **In Banach spaces:** A sequence x_n is said to converge weakly to an element x if $f(x_n) \to f(x)$ for any bounded linear functional f.
- **In probability theory:** Let us consider random variables $X_1, X_2, \ldots$, and let their cumulative distribution functions be $F_1, F_2, \ldots$, respectively. Then it is said that this sequence of variables converges weakly (or converges in distribution) to another random variable X (with cumulative distribution function F), if $\lim_{n\to\infty} F_n(x) = F(x)$ for every number $x \in \mathbb{R}$ at which F is continuous.

Another operation is the multiplication of a distribution with a function.

Definition 4.10. Let $\psi \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Then

$$(\psi u)(\varphi) := u(\psi\varphi).$$

Remark 4.3. If $u = T_f$ ($f \in L^1_{loc}(\Omega)$) and $\psi \in C^\infty$, then $\forall \varphi \in \mathcal{D}(\Omega)$:

$$(\psi T_f)(\varphi) = T_f(\psi\varphi) = \int_\Omega f\psi\varphi = \int_\Omega (f\psi)\varphi = T_{f\psi}(\varphi),$$

meaning that the multiplication defined above can be thought of as a generalization of the multiplication of functions.

4.4 Differentiation of distributions

The goal of the introduction of distributions is to somehow generalize L^1_{loc} functions (using regular distributions) in a way that their derivatives can be computed. Since generalizations should not change the notion we started from, our main goal here is to have an operator ∂ which acts similarly as the well-known derivative operator, namely we would like to have something like "$\partial^\alpha T_f = T_{\partial^\alpha f}$" (here f is sufficiently smooth) - this basically means the move of the derivative operator inside the integral.

Proposition 4.7. *Let $f \in C^1(\mathbb{R}^n)$. Then*

$$T_{\partial_j f}(\varphi) = -T_f(\partial_j \varphi), \qquad \forall \varphi \in C^1_0(\mathbb{R}^n).$$

Proof. (The same as the solution of Exercise 1 from Practice 5.)

Without loss of generality let us assume that $j = 1$ (the derivative is in the first variable), and let us use the following notation for $x \in \mathbb{R}^n$: $x = (x_1, \tilde{x})$, where $\tilde{x} \in \mathbb{R}^{n-1}$ (so here x_1 is the first variable, and $\tilde{x}$ contains all the other, $(n-1)$-many variables).

Then, for $f \in C^1(\mathbb{R}^n)$, f and $\partial_1 f$ are in $L^1_{loc}(\mathbb{R}^n)$, which means that the notations T_f and $T_{\partial_1 f}$ make sense. Then, by definition:

$$T_{\partial_1 f}(\varphi) = \int_{\mathbb{R}^n} (\partial_1 f)\varphi = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_1 f(x_1, \tilde{x})\varphi(x_1, \tilde{x}) dx_1 d\tilde{x} =$$

Now we apply the integration by parts formula for the inner integral:

$$= \int_{\mathbb{R}^{n-1}} [f(x_1, \tilde{x})\varphi(x_1, \tilde{x})]_{x_1=-\infty}^{\infty} d\tilde{x} - \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x_1, \tilde{x})\partial_1\varphi(x_1, \tilde{x}) dx_1 d\tilde{x} =$$

Now we use the fact that φ has compact support, so φ has zero values at $-\infty$ and at ∞, meaning that the first term is zero.

$$= -\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x_1, \tilde{x})\partial_1\varphi(x_1, \tilde{x}) dx_1 d\tilde{x} = -\int_{\mathbb{R}^n} f\partial_1\varphi = -T_f(\partial_1\varphi),$$

which gives our statement. □

Remark 4.4. A similar result can be stated on an arbitrary Ω domain (in that case, the Gauss-Ostrogradsky theorem should be used instead of partial integration).

Corollary 4.8. *Let $f \in C^m(\mathbb{R}^n)$ ($m \in \mathbb{N}$), and let α be a multiindex, $|\alpha| \leq m$. Then*

$$T_{\partial^\alpha f}(\varphi) = (-1)^{|\alpha|} T_f(\partial^\alpha \varphi).$$

Note that the term on the right-hand side does not have derivatives of f inside, meaning that this expression can also be used as a derivative of a not that smooth function (or more precisely, the derivative of the corresponding regular distribution).

Definition 4.11. Let $u \in \mathcal{D}'(\Omega)$. Then the $\partial_j u : \mathcal{D}(\Omega) \to \mathbb{R}$ **partial derivative** of u is defined as

$$\partial_j u(\varphi) := -u(\partial_j \varphi), \qquad \varphi \in \mathcal{D}(\Omega). \tag{4.7}$$

Proposition 4.9. *The functional $\partial_j u$ defined in* (4.7) *is a distribution.*

Proof. (The same as the solution of Exercise 2. from Practice 5.)

For $\partial_j u$ to be a distribution, we need two properties: it should be linear, and also sequentially continuous. The linearity is trivially true.

For the sequentially continuous property, let us assume that $\varphi_k \xrightarrow{\mathcal{D}(\Omega)} \varphi$. This means that then $\partial_j \varphi_k \to \partial_j \varphi$, and this also holds for all the derivatives of φ_k and φ; also, the supports are inside a compact set (because of the definition of $\varphi_k \xrightarrow{\mathcal{D}(\Omega)} \varphi$). So this means that $\partial_j \varphi_k \xrightarrow{\mathcal{D}(\Omega)} \partial_j \varphi$ also holds, and since u is a distribution, it is sequentially continuous, so $u(\partial_j \varphi_k) \to u(\partial_j \varphi)$ is also true, meaning that $u(\partial_j \varphi)$ is sequentially continuous, so it is a distribution. □

Now we define the general derivatives of a distribution:

Definition 4.12. Let $u \in \mathcal{D}'(\Omega)$ and let α be a multiindex. Then

$$\partial^\alpha u(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \qquad \varphi \in \mathcal{D}(\Omega).$$

Examples:

1. Let us consider the **Heaviside function**:

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Then, it can be shown that $T'_H(\varphi) = \delta_0(\varphi)$, or, if we use a shorter notation, $H' = \delta_0$. In this case H' means the derivative of this function in the distributional (or weak) sense.

Proof. (The same as the solution of Exercise 4. (c) from Practice 5.)

By definition:

$$T'_H(\varphi) = -T_H(\varphi') = -\int_{-\infty}^{\infty} H(x)\varphi'(x)dx = -\int_0^{\infty} \varphi'(x)dx$$

Now by the Newton-Leibniz formula (or the fundamental theorem of calculus):

$$= -[\varphi(x)]_0^\infty = \varphi(0) = \delta_0(\varphi),$$

where we used that φ has compact support, so its values at ∞ are zero. □

Remark 4.5. This statement is an easy corollary of the proposition stated in the next example.

2. **The derivative of a piecewise smooth functions:**

Proposition 4.10. *Let $(a, b) \subset \mathbb{R}$ be an interval (can be infinite), with*

$$a = a_0 < a_1 < a_2 < \cdots < a_m < a_{m+1} = b.$$

Suppose that we have a function $f : (a, b) \to \mathbb{R}$ which is continuously differentiable on the inside of these smaller intervals, meaning that $f \in C^1(a_k, a_{k+1})$ holds for $k = 0, 1, \ldots, m$, and also f has left $(f(a_k+0))$ and right $(f(a_k - 0))$ limits at points $a_1, a_2, \ldots a_m$. Also, suppose that $f' \in L^1_{loc}(a, b)$, where f' means the classical derivative of f taken on the inside of these small segments. Then

$$(T_f)' = T_{f'} + \sum_{k=1}^{m} (f(a_k + 0) - f(a_k - 0))\delta_{a_k},$$

in which $f(a_k + 0) - f(a_k - 0)$ means the jump at point a_k.

Proof. By the assumptions we know that $f \in L^1_{loc}(a, b)$, meaning that T_f exists. Then

$$T_f'(\varphi) = -T_f(\varphi') = -\int_a^b f\varphi' = -\sum_{k=0}^{m} \int_{a_k}^{a_{k+1}} f\varphi' =$$

By partial integration:

$$= -\sum_{k=0}^{m} \left([f\varphi]_{a_k}^{a_{k+1}} - \int_{a_k}^{a_{k+1}} f'\varphi \right) =$$

$$= (f\varphi)|_{a+0} + \sum_{k=0}^{m} (f(a_k + 0)\varphi(a_k) - f(a_k - 0)\varphi(a_k)) - (f\varphi)|_{b-0} + \int_a^b f'\varphi =$$

Since φ has compact support, we know that $(f\varphi)|_{a+0} = (f\varphi)|_{b-0} = 0$, meaning that

$$= \sum_{k=0}^{m} (f(a_k + 0) - f(a_k - 0))\, \varphi(a_k) + \int_a^b f'\varphi =$$

$$= \sum_{k=0}^{m} (f(a_k + 0) - f(a_k - 0))\, \delta_{a_k} + T_f(\varphi),$$

which was the statement we wanted to prove. □

3. **The solution of the wave equation in one dimension.**

Let us define the function

$$E(x_0, x_1) := \frac{1}{2} H(x_0 - |x_1|),$$

in which H is the Heaviside function. It is easy to see that $E \in L^1_{loc}(\mathbb{R})$.

Proposition 4.11. *The function E is a fundamental solution of the wave equation, meaning that*

$$\partial_0^2 T_E - \partial_1^2 T_E = \delta_0$$

For details, see Chapters 5 and 6.

4. **The solution of the heat equation in n dimensions.**
   Let us define the function

$$E(x_0, x) := \begin{cases} \dfrac{1}{(2\sqrt{\pi x_0})^n} \exp\left(\dfrac{-|x|^2}{4x_0}\right) & \text{if } x_0 > 0 \\ 0 & \text{if } x_0 \leq 0. \end{cases}$$

Proposition 4.12. *$E \in L^1_{loc}(\mathbb{R}^{n+1})$, and it is a fundamental solution of the heat equation, meaning that*

$$\partial_0^2 T_E - \partial_1^2 T_E = \delta_0$$

For details, see Chapters 5 and 7.

4.5 Cartesian product of distributions

As we will see in Chapter 5, the solutions of several PDEs can be written in the form $u_f * f$, where u_f is the fundamental solution of the equation (this will be defined later), f is the right-hand side of the equation (the 'source term'), and $*$ denotes the convolution of these distributions. To be able to define convolutions, we will need the definition of Cartesian products.

4.5.1 Definition

Before we define the Cartesian product of distributions, let us first consider functions.

Definition 4.13. Let $f \in L^1_{loc}(\mathbb{R}^n)$, $g \in L^1_{loc}(\mathbb{R}^m)$. Then **the Cartesian product of functions f and g** is defined as

$$(f \times g)(x, y) = f(x)g(y), \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

Remark 4.6. By Fubini's theorem, $f \times g \in L^1_{loc}(\mathbb{R}^{n+m})$.

Remark 4.7. Note that usually functions are defined (using set theory) as special relations, which means that they are actually just a collection of (usually infinitely many) ordered pairs (collecting the values x and $f(x)$). It is also possible to take the Cartesian products of such sets, but that would result in a different form than the one defined above, so we do not take that route.

Similarly as in the case of the derivatives of distributions, we will try to extend this definition (which only works for functions) to distributions using regular distributions. So let us consider $T_{f\times g}$, and see what happens!

Proposition 4.13. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, $g \in L^1_{loc}(\mathbb{R}^m)$. Then*

$$T_{f\times g}(\varphi) = T_f \left\{ x \longrightarrow T_g \left[y \longrightarrow \varphi(x, y) \right] \right\}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^{n+m}).$$

(Here the notation $y \longrightarrow \varphi(x, y)$ means that we think of $\varphi(x, y)$ as a one-variable function in y, and think of x as a parameter.)

Proof. By the definition of regular distributions:

$$T_{f\times g}(\varphi) = \int_{\mathbb{R}^{n+m}} (f \times g)\varphi =$$

Now we use the definition of $(f \times g)$:

$$= \int_{\mathbb{R}^{n+m}} f(x)g(y)\varphi(x,y)\ dx\ dy =$$

$$= \int_{\mathbb{R}^n} f(x) \left[\int_{\mathbb{R}^m} g(y)\varphi(x,y)\ dy \right] dx = T_f \left\{ x \longrightarrow T_g \left[y \longrightarrow \varphi(x,y) \right] \right\}.$$

□

Using the previous statement, our goal is to define the Cartesian product of distributions as

$$(u \times v)(\varphi) = u\{x \longrightarrow v[y \longrightarrow \varphi(x,y)]\}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^{n+m}). \tag{4.8}$$

The question is whether this definition makes sense. First of all, it is clear that $y \longrightarrow \varphi(x,y) \in \mathcal{D}(\mathbb{R}^m)$, meaning that $v[y \longrightarrow \varphi(x,y)]$ is well defined. The one remaining question is whether $x \longrightarrow v[y \longrightarrow \varphi(x,y)]$ is inside $\mathcal{D}(\mathbb{R}^n)$ or not. The next theorem answers this question.

Theorem 4.14. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^{n+m})$, and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is defined as*

$$\Psi(x) = v[y \longrightarrow \varphi(x,y)], \qquad (x \in \mathbb{R}^n).$$

Then $\Psi \in C_0^\infty(\mathbb{R}^n)$, and the $A : \mathcal{D}(\mathbb{R}^{n+m}) \to \mathcal{D}(\mathbb{R}^n)$ operator defined as

$$A(\varphi) = \Psi = v[y \longrightarrow \varphi(x,y)]$$

is linear and sequentially continuous (in the convergence defined on $\mathcal{D}(\mathbb{R}^{n+m})$ and $\mathcal{D}(\mathbb{R}^n)$).

Proof. We prove the theorem in four steps: in Step 1, we show that Ψ has compact support and it is continuous, then in Step 2 we prove its differentiability, and in Step 3 that it can be differentiated any number of times. In Step 4 it is proved that A is a distribution.

1. **Step 1:** By the definition of Ψ, it is easy to see that its support is a map of the compact set $\operatorname{supp}(\varphi)$ onto $\mathbb{R}^n$, so $\operatorname{supp}(\Psi)$ is compact. Now we prove that Ψ is continuous.

   Let us assume that for a sequence $(x_k) \in \mathbb{R}^n$ we have $x_k \to x$. We show that $\Psi(x_k) \to \Psi(x)$.

   For the sake of simplicity let us use the notation $\chi_k(y) = \varphi(x_k, y)$ and $\chi(y) = \varphi(x,y)$ (here x is fixed). It is enough to show that $\chi_k \xrightarrow{\mathcal{D}(\Omega)} \chi$, since then by the definition of Ψ and the (sequential) continuity of v we have

   $$\Psi(x_k) = v[y \longrightarrow \chi_k] \to v[y \longrightarrow \chi] = \Psi(x).$$

   It is evident that the supports of functions χ_k are part of the map of $\operatorname{supp}(\varphi)$ onto $\mathbb{R}^n$. Also, for any α multiindex $\partial_y^\alpha \chi_k(y) \to \partial_y^\alpha \chi(y)$ uniformly on $\mathbb{R}^n$, since by the theorem of Heine, a function φ with compact support is uniformly continuous, so

   $$|\partial_y^\alpha \chi_k(y) \to \partial_y^\alpha \chi(y)| = |\partial_y^\alpha \varphi(x_k, y) - \partial_y^\alpha \varphi(x_k, y)| < \varepsilon$$

   when $|x_k - x|$ is small enough. This means that we have proved that $\Psi \in C_0(\mathbb{R}^n)$.

2. **Step 2:** In this step we show that for $j = 1, \dots, n$ we have

$$\partial_j \Psi(x) = v[y \longrightarrow \partial^\alpha_{x_j} \varphi(x, y)],$$

from which we get that $\Psi \in C^1(\mathbb{R}^n)$.

Let $x \in \mathbb{R}^n$ be a fixed point, and let us choose an arbitrary real sequence (h_k) in a way that $h_k \to 0$ and $h_k \neq 0$. Let $h_k^{(j)} \in \mathbb{R}^n$ be such a vector which has zeros as all of its elements, except for the jth element which is h_k. Then

$$\begin{aligned} \frac{\Psi(x + h_k^{(j)}) - \Psi(x)}{h_k} &= \frac{1}{h_k} \Big(v[y \longrightarrow \varphi(x + h_k^{(j)}, y)] - [y \longrightarrow \varphi(x, y)] \Big) = \\ &= v \left[y \to \frac{\varphi(x + h_k^{(j)}, y) - \varphi(x, y)}{h_k} \right]. \end{aligned} \tag{4.9}$$

By the Lagrange intermediate value theorem, there exists a real number $0 < \theta_k < 1$ in a way that

$$\varphi(x + h_k^{(j)}, y) - \varphi(x, y) = h_k \partial_{x_j} \varphi(x + \theta_k h_k^{(j)}, y),$$

which means that by (4.9) we have

$$\frac{\Psi(x + h_k^{(j)}) - \Psi(x)}{h_k} = v \left[y \to \partial_{x_j} \varphi(x + \theta_k h_k^{(j)}, y) \right] = v \left[y \to \kappa_k(y) \right], \tag{4.10}$$

in which $\kappa_k(y) = \partial_{x_j} \varphi(x + \theta_k h_k^{(j)}, y)$. Then similarly as in Step 1, we can prove that $\kappa_k \xrightarrow{\mathcal{D}(\Omega)} \kappa$, where $\kappa = \partial_{x_j} \varphi(x, y)$. Then by the (sequential) continuous property of v and by (4.10), we have

$$\lim_{k \to \infty} \frac{\Psi(x + h_k^{(j)}) - \Psi(x)}{h_k} = v[y \longrightarrow \partial^\alpha_{x_j} \varphi(x, y)],$$

which was the statement we wanted to prove.

3. **Step 3:** By induction it can be proved using the results of Step 2 that

$$\partial^\alpha \Psi(x) = v \left[y \to \partial^\alpha_x \varphi(x, y) \right],$$

where α is an arbitrary multiindex. Then $\Psi \in C_0^\infty(\mathbb{R}^n)$.

4. **Step 4:** We show here that A is sequentially continuous (the linear property is trivial). Our goal here is to show that if there is a sequence for which $\varphi_k \xrightarrow{\mathcal{D}(\mathbb{R}^{n+m})} \varphi$, then we have $A(\varphi_k) \xrightarrow{\mathcal{D}(\mathbb{R}^n)} A(\varphi)$. By the convergence of the functions φ_k we know that there is a compact set $K_0 \subset \mathbb{R}^{n+m}$ in a way that $\operatorname{supp}(\varphi_k) \subset K_0$. Let K be the image of the set K_0 on $\mathbb{R}^n$: then K is compact and $\operatorname{supp}(\psi_k) \subset K_0$, in which $\psi_k(y) = \varphi_k(x, y)$ (x is fixed). Since

$$A(\varphi_k) = v(\psi_k) = v[y \to \varphi(x, y)],$$

then $\operatorname{supp}(A\varphi_k) \subset K_0$.

We also have to show that for all α multiindexes $\partial^\alpha(A\varphi_k) \to \partial^\alpha(A\varphi)$ uniformly. Then by Step 3, we have

$$\partial^\alpha(A\varphi_k)(x) = v \left[y \to \partial^\alpha_x \varphi_k(x, y) \right],$$

then

$$|\partial^\alpha(A\varphi_k)(x) - \partial^\alpha(A\varphi)(x)| = |v \left[y \to \partial^\alpha_x \varphi_k(x, y) - \partial^\alpha_x \varphi(x, y) \right]| \leq$$

Now we use Theorem 4.1:

$$\leq C_{K_0} \sum_{|\beta| \leq m_{K_0}} \sup_{y \in K_0} \left| \partial^\alpha_x \partial^\beta_y \varphi_k(x, y) - \partial^\alpha_x \partial^\beta_y \varphi(x, y) \right| \leq$$

$$\leq C_{K_0} \sum_{|\beta| \leq m_{K_0}} \sup_{(x,y) \in K} \left| \partial^\alpha_x \partial^\beta_y \varphi_k(x, y) - \partial^\alpha_x \partial^\beta_y \varphi(x, y) \right| \to 0$$

where we used that the sequence φ_k converges in the $\mathcal{D}(\mathbb{R}^n)$-sense. Then it means that $(\partial^\alpha(A\varphi_k) - \partial^\alpha(A\varphi)) \to 0$ uniformly.

This concludes the proof. □

Theorem 4.14. states that the term (4.8) makes sense, since the function $x \longrightarrow v[y \longrightarrow \varphi(x,y)]$ is inside $\mathcal{D}(\mathbb{R}^n)$, and the functional $u\{x \longrightarrow v[y \longrightarrow \varphi(x,y)]\}$ is indeed a distribution. Thus, the Cartesian product of distributions can be formulated in the following way.

Definition 4.14. Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in \mathcal{D}'(\mathbb{R}^m)$. Then the **Cartesian product of distributions u and v** is defined as

$$(u \times v)(\varphi) := u\{x \longrightarrow v[y \longrightarrow \varphi(x,y)]\}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^{n+m}).$$

Remark 4.8. By Proposition 4.13, $T_{f\times g} = T_f \times T_g$.

4.5.2 Properties of the Cartesian product

Proposition 4.15 (Commutativity)**.** *Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in \mathcal{D}'(\mathbb{R}^m)$. Then*

$$(u \times v)(\varphi) = (v \times u)(\varphi). \tag{4.11}$$

The proof was skipped in 2024.

Proof. For the proof, we are going to use the following lemma:

Lemma 4.16. *Let us define a set of functions the following way:*

$$\varphi(x,y) := \sum_{j=1}^{N} \psi_j(x)\chi_j(y) \tag{4.12}$$

in which $\psi_j \in \mathcal{D}(\mathbb{R}^n)$, $\chi_j \in \mathcal{D}(\mathbb{R}^m)$, and N is arbitrary. Then the set of functions in the form (4.12) *are dense in $\mathcal{D}(\mathbb{R}^{n+m})$ (in the convergence defined before).*

Proof. (Lemma 4.16) The thing we have to prove here is that there is a sequence φ_k for which all of its elements are in the form (4.12) and $\varphi_k \xrightarrow{\mathcal{D}(\mathbb{R}^{n+m})} \varphi$.

Let us assume that

$$\operatorname{supp}(\varphi) \subset K_a^{n+m}(0) = \left\{(x,y) \in \mathbb{R}^{n+m} : |x_j| < a, |y_l| < a\right\}.$$

Then the Fourier series of function φ converges uniformly on $K_{2a}^{n+m}(0)$ to function φ in the set of functions defined as

$$(x,y) \to (4a)^{-\frac{n+m}{2}} \exp\left(\frac{\pi}{2a} i\left[\langle \beta, x\rangle + \langle \gamma, x\rangle\right]\right),$$

in which $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ are vectors with elements as whole numbers. Also, by differentiating partially the Fourier series, the sequence of the derivatives also converges to $\partial^\alpha \varphi$. This means that

$$\varphi(x,y) = \sum_{l=0}^{\infty} \sum_{|\beta|+|\gamma|=1} c_{\beta,\gamma}(4a)^{-\frac{n+m}{2}} \exp\left(\frac{\pi}{2a} i\left[\langle \beta, x\rangle + \langle \gamma, x\rangle\right]\right)$$

Let us consider the partial sums of this infinite series:

$$\tilde{\varphi}_k(x,y) = \sum_{l=0}^{k} \sum_{|\beta|+|\gamma|=1} c_{\beta,\gamma}(4a)^{-\frac{n+m}{2}} \exp\left(\frac{\pi}{2a} i\left[\langle \beta, x\rangle + \langle \gamma, x\rangle\right]\right) =$$

$$= \sum_{l=0}^{k} \sum_{|\beta|+|\gamma|=1} c_{\beta,\gamma}(4a)^{-\frac{n+m}{2}} \exp\left(\frac{\pi}{2a} i\,\langle \beta, x\rangle\right) \cdot \exp\left(\frac{\pi}{2a} i\,\langle \gamma, x\rangle\right) = \sum_{j=1}^{N} \tilde{\psi}_j(x)\tilde{\chi}_j(y)$$

Functions $\tilde{\varphi}_k$ are in the form (4.12), but we only have $\tilde{\psi}_j \in C^\infty(\mathbb{R}^n)$ and $\tilde{\chi}_j \in C^\infty(\mathbb{R}^m)$. We also know that for the sequence $\tilde{\varphi}_k$ it derivatives are also uniformly convergent.

Now we construct the sequence φ_k. Let $f \in C_0^\infty(K_{2a}^n(0))$ and $g \in C_0^\infty(K_{2a}^m(0))$ such functions which equal to zero on a neighborhood of the squares $K_a^n(0)$ and $K_a^m(0)$, respectively. Then we define functions φ_k as

$$\varphi_k(x,y) = f(x)g(y)\tilde{\varphi}_k(x,y) = \sum_{j=1}^{N} \left[f(x)\tilde{\psi}_j(x)\right] \left[g(y)\tilde{\chi}_j(y)\right].$$

Then it is clear that these functions are in the form (4.12). We then show that $(\varphi_k) \xrightarrow{\mathcal{D}(\mathbb{R}^{n+m})} \varphi$. Since $\operatorname{supp}(\varphi) \subset K_{2a}^{n+m}(0)$, it is enough to prove that $\lim(\partial^{\alpha}\varphi_k) = \partial^{\alpha}\varphi$ uniformly.

If $(x,y) \in K_a^{n+m}(0)$, then

$$|\partial^{\alpha}\varphi_k(x,y) - \partial^{\alpha}\varphi(x,y)| = |\partial^{\alpha}[fg\tilde{\varphi}_k](x,y) - \partial^{\alpha}\varphi(x,y)| = |\partial^{\alpha}\tilde{\varphi}_k(x,y) - \partial^{\alpha}\varphi(x,y)|.$$

If $(x,y) \notin K_a^{n+m}(0)$, then by the Leibniz rule (Theorem 3.1) we have

$$|\partial^{\alpha}\varphi_k(x,y) - \partial^{\alpha}\varphi(x,y)| = |\partial^{\alpha}\varphi_k(x,y)| = |\partial^{\alpha}[fg\tilde{\varphi}_k](x,y)| =$$

$$= \left|\sum_{\tilde{\alpha}\leq\alpha} d_{\tilde{\alpha}}\partial^{\alpha-\tilde{\alpha}}(fg)(x,y)\partial^{\tilde{\alpha}}\tilde{\varphi}_k(x,y)\right| \leq c\sum_{\tilde{\alpha}\leq\alpha}\left|\partial^{\tilde{\alpha}}\tilde{\varphi}_k(x,y)\right| = c\sum_{\tilde{\alpha}\leq\alpha}\left|\partial^{\tilde{\alpha}}\tilde{\varphi}_k(x,y) - \partial^{\tilde{\alpha}}\varphi(x,y)\right|.$$

Since $\lim(\partial^{\alpha}\tilde{\varphi}_k) = \partial^{\alpha}\varphi$ uniformly, then we have the uniform convergence of $\partial^{\alpha}\varphi_k$ too, which concludes this proof. □

(Continuation of the proof of Proposition 4.15.)

We prove Proposition 4.15 in two steps: first we prove property (4.11) for functions in the form (4.12), then in step 2 (using the fact that functions in the form (4.12) are dense) we prove it for any arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^{n+m})$.

Step 1: Let us assume that our φ function is in the form (4.12), i.e.

$$\varphi(x,y) = \sum_{j=1}^{N} \psi_j(x)\chi_j(y).$$

Then substituting it into the left-hand side of (4.11), we get:

$$(u \times v)(\varphi) = u\left\{x \longrightarrow v\left[y \longrightarrow \sum_{j=1}^{N}\psi_j(x)\chi_j(y)\right]\right\} =$$

Now we use the fact that v only acts in variable y (and $\psi_j(x)$ does not depend on y).

$$= u\left\{x \longrightarrow \sum_{j=1}^{N}\psi_j(x)v(\chi_j(y))\right\} =$$

Similarly, u only depends on x, meaning that

$$= \sum_{j=1}^{N} u(\psi_j(x))v(\chi_j(y)).$$

By the same arguments, for the right-hand side of (4.11) we get:

$$(v \times u)(\varphi) = v\left\{y \longrightarrow u\left[x \longrightarrow \sum_{j=1}^{N}\psi_j(x)\chi_j(y)\right]\right\} =$$

$$= v\left\{x \longrightarrow \sum_{j=1}^{N} u(\psi_j(x))\chi_j(y)\right\} = \sum_{j=1}^{N} u(\psi_j(x))v(\chi_j(y)).$$

So they are equal, meaning that we proved the statement for functions in the form (4.12).

Step 2: Let $\varphi \in \mathcal{D}(\mathbb{R}^{n+m})$ be arbitrary. Then by Lemma 4.16 there is a sequence $\{\varphi_k\} \subset \mathcal{D}(\mathbb{R}^{n+m})$, for which all of the elements are in the form (4.12), and $\varphi_k \xrightarrow{\mathcal{D}(\mathbb{R}^{n+m})} \varphi$. Then because of Step 1,

$$(u \times v)(\varphi_k) = u\{x \longrightarrow v[y \longrightarrow \varphi_k(x,y)]\} =$$

$$= v\{y \longrightarrow u[x \longrightarrow \varphi_k(x,y)]\} = (v \times u)(\varphi_k).$$

Here $(u \times v)(\varphi_k)$ is a distribution defined on $\mathcal{D}(\mathbb{R}^{n+m})$, meaning that by the sequentially continuous property, if $\varphi_k \xrightarrow{\mathcal{D}(\mathbb{R}^{n+m})} \varphi$, then $(u \times v)(\varphi_k) \to (u \times v)(\varphi)$. Similarly, for the right-hand side we get $(v \times u)(\varphi_k) \to (v \times u)(\varphi)$, which means that $(u \times v)(\varphi) = (v \times u)(\varphi)$, and since φ was arbitrary here, we get property (4.11). □

Proposition 4.17 (Linearity)**.** *Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $v_1, v_2 \in \mathcal{D}'(\mathbb{R}^m)$, λ_1, $\lambda_2 \in C^\infty(\mathbb{R}^m)$. Then*

$$u \times (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1(u \times v_1) + \lambda_2(u \times v_2).$$

Proof. By definition,

$$u \times (\lambda_1 v_1 + \lambda_2 v_2)(\varphi) = u\{x \longrightarrow (\lambda_1 v_1 + \lambda_2 v_2)[y \longrightarrow \varphi(x,y)]\} =$$

$$= u\{x \longrightarrow ((\lambda_1 v_1)[y \longrightarrow \varphi(x,y)] + (\lambda_2 v_2)[y \longrightarrow \varphi(x,y)])\} =$$

$$= u\{x \longrightarrow (\lambda_1 v_1)[y \longrightarrow \varphi(x,y)]\} + u\{x \longrightarrow (\lambda_2 v_2)[y \longrightarrow \varphi(x,y)]\} =$$

$$= \lambda_1(u \times v_1) + \lambda_2(u \times v_2),$$

which concludes the proof. □

Proposition 4.18. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^m)$. Then for all α multi-index*

$$\partial_y^\alpha(u \times v) = u \times \partial^\alpha v,$$

and

$$\partial_x^\alpha(u \times v) = \partial^\alpha u \times v.$$

Proof. By definition,

$$\partial_y^\alpha(u \times v)(\varphi) = (-1)^{|\alpha|}(u \times v)(\partial_y^\alpha \varphi) = (-1)^{|\alpha|} u\left\{x \to v\left[y \to \partial_y^\beta \varphi(x,y)\right]\right\} =$$

$$= u\left\{x \to \partial_y^\beta v\left[y \to \varphi(x,y)\right]\right\} = \left(u \times (\partial_y^\beta v)\right)(\varphi).$$

The other part can be proved similarly. □

Proposition 4.19 (Support)**.** *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^m)$. Then*

$$\operatorname{supp}(u \times v) = \operatorname{supp}(u) \times \operatorname{supp}(v).$$

Proof. We prove that the sets are parts of each other.

Let us assume proceeding towards contradiction that for some point $(x,y) \notin \operatorname{supp}(u) \times \operatorname{supp}(v)$, but $(x,y) \in \operatorname{supp}(u \times v)$. Let us assume that $x \notin \operatorname{supp} u$. Then there is a neighborhood U_x and $\forall \varphi \in \mathcal{D}(\Omega_1)$ for which $\operatorname{supp}\varphi \subset U_x$ we have $u(\varphi) = 0$. We know that there is some $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ function for which $\operatorname{supp}\varphi \subset U_x \times \Omega_2$. Also, the support of $x \mapsto [v(y \mapsto \varphi(x,y))]$ is in U_x, but by the previous arguments $u\{x \mapsto [v(y \mapsto \varphi(x,y))]\} = 0$ if $x \in U_x$ (y is arbitrary). Then $(x,y) \notin \operatorname{supp}(u \times v)$, which is a contradiction.

For the other implication let us assume that $(x,y) \notin \operatorname{supp}(u \times v)$, but $(x,y) \in \operatorname{supp}(u) \times \operatorname{supp}(v)$. Then $\exists U \subset \Omega_1$ ($x \in U$) and $\varphi_1 \in \mathcal{D}(\Omega_1)$ for which $\operatorname{supp}\varphi_1 \subset U$. Also, $\exists V \subset \Omega_2$ ($y \in V$) and $\varphi_2 \in \mathcal{D}(\Omega_2)$, for which $\operatorname{supp}\varphi_2 \subset V$ in a way that $(U \times V) \cap \operatorname{supp}(u \times v) \neq \emptyset$, and let$u(\varphi_1) = c_1$ and $v(\varphi_2) = c_2$ ($c_1, c_2 \neq 0$). Now we define a function $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ in a way that $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ and $\operatorname{supp}\varphi \subset U \times V$. By using the definitions:

$$0 = (u \times v)\varphi(x,y) = u\{x \mapsto [v(y \mapsto \varphi_1\varphi_2)]\} = u\{x \mapsto \varphi_1(x)v(\varphi_2)\} = c_2 u(\varphi_1(x)) = c_1 c_2 \neq 0,$$

so we get a contradiction. □

4.6 Convolution of distributions

As mentioned at the beginning of this Section 4.5, we will see in Chapter 5 that the solutions of certain PDEs are in the form $u_f * f$, in which $*$ is the convolution of these two distributions. First, we define the convolutions of two functions (or remind you of the definition, since I am pretty sure you have learned it in some other classes), and then we generalize it to distributions.

4.6.1 Convolution of functions

Definition 4.15. Let $f, g \in L^1_{loc}(\mathbb{R}^n)$, and suppose that the function

$$x \longrightarrow \int_{\mathbb{R}^n} |f(y)g(x-y)|dy \tag{4.13}$$

is in $L^1_{loc}(\mathbb{R}^n)$. Then we say that **the convolution of functions f and g** exists, and is defined as

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)dy, \qquad (x \in \mathbb{R}^n) \tag{4.14}$$

Remark 4.9. Because of condition (4.13), the integral inside (4.14) is finite for a.e. $x \in \mathbb{R}^n$. Then $f * g \in L^1_{loc}(\mathbb{R}^n)$, and if $f * g$ exists, then $g * f$ also exists, and $f * g = g * f$.

A problem here is that condition (4.13) is hard to check. Because of this, in practice people use several other conditions which imply (4.13), e.g. the next one, which is perhaps the most well-known.

Proposition 4.20. *Let $f, g \in L^1(\mathbb{R}^n)$. Then $f*g$ exists, and is inside $L^1(\mathbb{R}^n)$.*

Proof. We apply Fubini's theorem to the non-negative function $(x, y) \longrightarrow |f(y)g(x-y)|$:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)g(x-y)|dydx = \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x-y)|dxdy \leq$$

$$\leq \int_{\mathbb{R}^n} |f(y)| \, \|g\|_{L^1(\mathbb{R}^n)}dy \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

Because of this, the function $x \longrightarrow \int_{\mathbb{R}^n} |f(y)g(x-y)|dy$ is inside $L^1(\mathbb{R}^n)$, meaning that $f * g$ exists, and it is also inside $L^1(\mathbb{R}^n)$. □

As mentioned before, there are several other conditions under which $f * g$ exists, e.g. it is enough that one of the functions is zero outside a compact set.

4.6.2 Convolution of distributions

Similarly as in the previous section, and also in the case of the derivatives of distributions, we start off by examining the regular distributions, more precisely the regular distribution associated with $f * g$, and then we try to generalize the form of this object to get the general definition of the convolution of any two distributions.

Proposition 4.21. *Let $f, g \in L^1_{loc}(\mathbb{R}^n)$, and suppose that $f * g$ exists. Then*

$$T_{f*g}(\varphi) = \int_{\mathbb{R}^{2n}} f(y)g(z)\varphi(y+z)dydz, \qquad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{4.15}$$

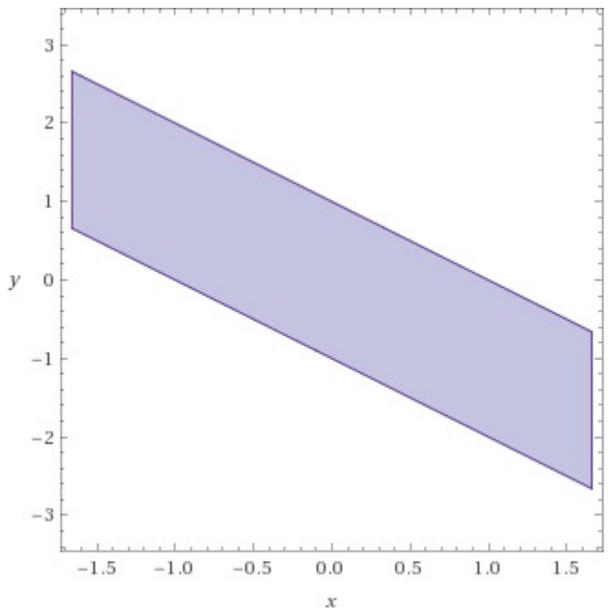


Figure 4.1: The infinite strip $|y+z| \leq 1$ (it continues on the left and on the right).

Proof. By definition,

$$T_{f*g}(\varphi) = \int_{\mathbb{R}^n} (f*g)\varphi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(x-y)\,dy\,\varphi(x)dx =$$

$$= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} g(x-y)\varphi(x)dxdy =$$

Now we define a new variable $z = x-y$, and substitute it to the inner integral:

$$= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} g(z)\varphi(y+z)dzdy = \int_{\mathbb{R}^{2n}} f(y)g(z)\varphi(y+z)dydz,$$

which completes the proof. □

Seemingly what we got is that

$$T_{f*g}(\varphi) \approx (T_f \times T_g)[(y,z) \longrightarrow \varphi(y+z)].$$

The problem here is that $\varphi(y+z)$ might not have compact support!

Example 4.1. Let φ be such a test function for which $\operatorname{supp}(\varphi) = B(0,1)$ (which is the sphere with radius one centered at the origin). Then

$$\operatorname{supp}\left[(y,z) \to \varphi(y+z)\right] = \left\{(y,z) \in \mathbb{R}^{2n} : |y+z| \leq 1\right\},$$

which is an infinite strip (see Figure 4.1), meaning that it is clearly not bounded, so it is not compact.

The way we are going to solve this problem in the following pages is to somehow "chop off" the tails of such functions, meaning that we will approximate these problematic functions with those which have compact support.

First we define a set of functions which will tend to the constant one function in some sense.

Definition 4.16. Let $\zeta_k \in C_0^\infty(\mathbb{R}^{2n}), \forall k$ a sequence of functions. Then we say that $\boldsymbol{\zeta_k \xrightarrow{(*)} 1}$ (or: $\boldsymbol{\zeta_k}$ **has the** $\boldsymbol{(*)}$**-property**), if the following two conditions hold:

(i) $\forall\alpha$ multi-index, $\partial^\alpha(\zeta_k - 1) \to 0$ uniformly on all compact subsets of $\mathbb{R}^{2n}$.

(ii) $\forall\alpha$ multi-index, there is a constant c_α such that

$$\sup_{k\in\mathbb{N}} \sup_{\mathbb{R}^{2n}} |\partial^\alpha \zeta_k| \le c_\alpha.$$

A natural question is whether such functions exist.

Remark 4.10. Let us consider such functions in $C_0^\infty(\mathbb{R}^{2n})$ for which

$$\begin{cases} \zeta(y,z) & = 1, \qquad \text{if } |(y,z)| \le 1, \\ |\zeta(y,z)| & \le 1, \qquad \text{otherwise.} \end{cases}$$

Such functions exist: see Section 3.4. Then $\zeta_k(y,z) := \zeta\left(\frac{y}{k}, \frac{z}{k}\right)$ are such functions for which $\zeta_k \xrightarrow{(*)} 1$.

Our goal now is that by using such sequences of functions, we approximate the problematic $\varphi(y+z)$ term inside (4.15). First we show that it works for regular distributions.

Proposition 4.22. *Let $f, g \in L^1_{loc}(\mathbb{R}^n)$, and suppose that $f * g$ exists. If $\zeta_k \xrightarrow{(*)} 1$ holds, then*

$$T_{f*g}(\varphi) = \int_{\mathbb{R}^{2n}} f(y)g(z)\varphi(y+z)dydz = \lim_{k\to\infty} \int_{\mathbb{R}^{2n}} f(y)g(z)\zeta_k(y,z)\varphi(y+z)dydz. \tag{4.16}$$

Proof. Since $\zeta_k \xrightarrow{(*)} 1$, then by definition $(\zeta_k - 1) \to 0$ uniformly on all compact subsets, so $\zeta_k \to 1$ point-wise in $\mathbb{R}^{2n}$. We also know that $\sup_{k\in\mathbb{N}} \sup_{\mathbb{R}^{2n}} |\partial^\alpha \zeta_k| \le c_\alpha$, so ζ_k is also uniformly bounded. Then by Lebesgue's theorem[5] we get that the right-hand side of (4.16) indeed converges to the left-hand side. □

Remark 4.11. The main advantage of this construction is of course that the function $\zeta_k(y,z)\varphi(y+z)$ now has a compact support.

Using the previous results, we can finally define the convolution of distributions.

Definition 4.17. Let us suppose that $u, v \in \mathcal{D}'(\mathbb{R}^n)$, $\zeta_k \xrightarrow{(*)} 1$, and also that the limit

$$\lim_{k\to\infty} (u \times v)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] \tag{4.17}$$

exists and is finite for $\forall\varphi \in \mathcal{D}(\mathbb{R}^n)$, and this limit depends continuously on φ[6]. Then we say that **the convolution of distributions u and v** exists, and is defined as

$$(u * v)(\varphi) = \lim_{k\to\infty} (u \times v)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] \qquad (\varphi \in \mathcal{D}(\mathbb{R}^n)). \tag{4.18}$$

Remark 4.12. Some remarks:

[5] **Lebesgue's theorem:** Suppose that $f_j : \Omega \to \mathbb{R}$ measurable functions, and $f_j \to f$ a.e. on Ω. Also suppose that there is such a function $g \in L^1(\Omega)$, for which $|f_j| \le g$ a.e. on Ω, $\forall j$. Then $f_j \to f$ in the L^1-norm, meaning that $\int_\Omega f_j \to \int_\Omega f$.

[6] Here *continuous dependence* means that if we change the function φ a little, i.e. $\varphi_\varepsilon = \varphi + \varepsilon$ $(\varepsilon \in \mathbb{R}^+)$, then the change of the value of $(u * v)(\varphi)$ should be bounded by ε, i.e. $\|(u * v)(\varphi) - (u * v)(\varphi_\varepsilon)\| \le \varepsilon$.

- Note that the convolution is defined only for distributions acting on the same space, i.e. if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^m)$, then the convolution is only defined if $n = m$. (The reason for this is that the term $y + z$ inside φ should make sense.)
- The limit (4.17) does not depend on the choice of ζ_k, i.e. it is the same for any sequence $\zeta_k \overset{(*)}{\longrightarrow} 1$.
- The operator defined as (4.18) is linear in φ.
- The assumption of continuous dependence can be omitted, and actually can be proved from the other properties: the general proof of this statement is hard, but for some given examples it might be easy.

Proposition 4.23. *If $f, g \in L^1_{loc}(\mathbb{R}^n)$, and $f * g$ exists, then $T_f * T_g$ also exists and $T_f * T_g = T_{f*g}$.*

This is an easy consequence of the previous proposition.

4.6.3 Properties of convolutions

In this section we list some useful properties of convolutions, which make their calculation much more easier. Most of them can be proved easily from the definition of the convolution.

Proposition 4.24. *Let us suppose that $u \in \mathcal{D}'(\mathbb{R}^n)$ and δ_0 is the Dirac-delta concentrated on zero. Then*

$$(u * \delta_0)(\varphi) = u(\varphi).$$

This means that the Dirac-delta is the identity element in the algebraic structure of convolutions.

Proof. (Proposition 4.24)

$$(u * \delta_0)(\varphi) = \lim_{k\to\infty} (u \times \delta_0)\left[(y, z) \to \zeta_k(y, z)\varphi(y + z)\right] =$$

$$= \lim_{k\to\infty} u\left\{y \longrightarrow \delta_0\left[z \longrightarrow \zeta_k(y, z)\varphi(y + z)\right]\right\} = \lim_{k\to\infty} u\left\{y \longrightarrow \zeta_k(y, 0)\varphi(y)\right\} =$$

$$= u\left\{y \longrightarrow \varphi(y)\right\} = u(\varphi),$$

where we used that ζ_k has the $(*)$-property. □

Proposition 4.25 (Commutative property)**.** *Let us suppose that $u, v \in \mathcal{D}'(\mathbb{R}^n)$ and $u * v$ exists. Then $v * u$ also exists, and $u * v = v * u$.*

Remark 4.13. An important remark here is that the associative property does not hold: it can be shown that for $u(\varphi) = T_H(\varphi)$ (the regular distribution corresponding to the Heaviside function), $v(\varphi) = \delta_0'(\varphi)$ (the derivative of the Dirac-delta) and $w(\varphi) = T_1(\varphi)$ (the regular distribution corresponding to the constant one function), we have

$$T_1 = (u * v) * w \neq u * (v * w) = T_0.$$

This means that the convolution operation defined on the set of distributions results in an algebraic structure which is commutative, but not associative. (A non-associative structure is called *magma* in algebra, and another example of such structure is the game "rock-paper-scissors", which is commutative, but not associative.)

Proof. (Proposition 4.25)

$$(u \times v)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] = u\left\{y \longrightarrow v\left[z \longrightarrow \zeta_k(y,z)\varphi(y+z)\right]\right\} =$$

By Proposition 4.15:

$$= v\left\{z \longrightarrow u\left[y \longrightarrow \zeta_k(y,z)\varphi(y+z)\right]\right\} =$$

By introducing new variables $\tilde{y} := z$ and $\tilde{z} := y$:

$$= v\left\{\tilde{y} \longrightarrow u\left[\tilde{z} \longrightarrow \zeta_k(\tilde{z},\tilde{y})\varphi(\tilde{z}+\tilde{y})\right]\right\} = (v \times u)\left[(\tilde{z},\tilde{y}) \to \zeta_k(\tilde{z},\tilde{y})\varphi(\tilde{y}+\tilde{z})\right]$$

If $\zeta_k \overset{(*)}{\longrightarrow} 1$, then for the sequence $\tilde{\zeta}_k(y,z) := \zeta_k(\tilde{z},\tilde{y})$ we also know that $\tilde{\zeta}_k \overset{(*)}{\longrightarrow} 1$, meaning that if $k \to \infty$, then $(u \times v)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right]$ tends to $(u * v)(\varphi)$, while $(v \times u)\left[(\tilde{z},\tilde{y}) \to \zeta_k(\tilde{z},\tilde{y})\varphi(\tilde{y}+\tilde{z})\right]$ tends to $(v*u)(\varphi)$, meaning that $(u*v)(\varphi) = (v*u)(\varphi)$. □

Proposition 4.26 (Linearity)**.** *Let us suppose that $u_1, u_2, v \in \mathcal{D}'(\mathbb{R}^n)$ and $u_1 * v, u_2 * v$ exists. Then for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we get*

$$(\lambda_1 u_1 + \lambda_2 u_2) * v = \lambda_1(u_1 * v) + \lambda_2(u_2 * v).$$

Proof. By definition,

$$\left[(\lambda_1 u_1 + \lambda_2 u_2)\right] * v = \lim_{k\to\infty}\left((\lambda_1 u_1 + \lambda_2 u_2) \times v\right)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] =$$

By Proposition 4.17,

$$= \lim_{k\to\infty}\left(\lambda_1(u_1 \times v) + \lambda_2(u_2 \times v)\right)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] = \lambda_1(u_1 * v) + \lambda_2(u_2 * v),$$

which gives the statement. □

Proposition 4.27 (Support)**.** *Let us suppose that $u, v \in \mathcal{D}'(\mathbb{R}^n)$ and $u * v$ exists. Then*

$$\operatorname{supp}(u * w) \subset \overline{\operatorname{supp}(u) + \operatorname{supp}(v)},$$

in which $\operatorname{supp}(u) + \operatorname{supp}(v) := \{y + z \in \mathbb{R}^n : y \in \operatorname{supp}(u), z \in \operatorname{supp}(u)\}$*, and the "line over the set" notation means the closure of that set.*

Proof. We prove this statement in the indirect way.

Let us assume that $x \in \operatorname{supp}(u * v)$, but $x \notin \overline{\operatorname{supp}\, u + \operatorname{supp}\, v}$. Then there is a neighborhood U_x, for which $U_x \cap \overline{\operatorname{supp}\, u + \operatorname{supp}\, v} = \emptyset$. Let $\varphi \in \mathcal{D}(\mathbb{R}^{2n})$, for which $\operatorname{supp}(\varphi) \subset U_x$. Then

$$(u * v)(\varphi) = \lim_{k\longrightarrow\infty}(u \times v)((y,z) \longrightarrow \varphi(y+z)\zeta_k(y,z)).$$

We know that if $y \in \operatorname{supp}(u)$, $z \in \operatorname{supp}(v)$, then

$$y + z \in \operatorname{supp}(u) + \operatorname{supp}(v) \subset \mathbb{R}^n \setminus U_x \subset \mathbb{R}^n \setminus \operatorname{supp}(\varphi),$$

meaning that the term $(y,z) \longrightarrow \varphi(y+z)\zeta_k(y,z)$ in the formula above is constant zero (in a neighborhood around the previously defined point (y,z)), so

$$\operatorname{supp}[(y,z) \longrightarrow \zeta_k(y,z)\varphi(y+z)] \subset \mathbb{R}^{2n} \setminus (\operatorname{supp}(u) \times \operatorname{supp}(v)),$$

but we know that $\operatorname{supp}(u \times v) = \operatorname{supp}(u) \times \operatorname{supp}(v)$ (Proposition 4.19), meaning that

$$(u \times v)[(y,z) \longrightarrow \zeta_k(y,z)\varphi(y+z)] = 0,$$

so $(u * v)(\varphi) = 0$, which measn that $x \notin \operatorname{supp}\,(u * v)$, which is a contradiction. □

Proposition 4.28 (Derivative)**.** *Let us suppose that $u, v \in \mathcal{D}'(\mathbb{R}^n)$ and $u * v$ exists. Then for all α multi-indexes*

$$\partial^\alpha(u * v) = (\partial^\alpha u) * v = u * (\partial^\alpha v).$$

Proof. It is enough to prove that for all $j = 1, 2, \dots n$ we have

$$\partial_j(u * v) = \partial_j u * v = u * \partial_j v,$$

since then the original statement can be proved by induction.

By definition,

$$\begin{aligned} \partial_j(u * v)(\varphi) = -(u * v)(\partial_j \varphi) = -\lim_{k\to\infty} (u \times v)\left[(y,z) \to \zeta_k(y,z)\partial_j\varphi(y+z)\right] = \\ = -\lim_{k\to\infty} u\left\{y \longrightarrow v\left[z \longrightarrow \zeta_k(y,z)\partial_j\varphi(y+z)\right]\right\}. \end{aligned} \tag{4.19}$$

Then by the properties of differentiation,

$$\zeta_k(y,z)\partial_j\varphi(y+z) = \zeta_k(y,z)\partial_{z_j}\varphi(y+z) =$$

$$= \partial_{z_j}\left(\zeta_k(y,z)\varphi(y+z)\right) - \partial_{z_j}(\zeta_k(y,z))\varphi(y+z).$$

By substituting this into (4.19), we get

$$\begin{aligned} &\partial_j(u * v)(\varphi) = \\ &\lim_{k\to\infty}\left(-u\left\{y \longrightarrow v\left[z \longrightarrow \partial_{z_j}\left(\zeta_k(y,z)\varphi(y+z)\right)\right]\right\} + u\left\{y \longrightarrow v\left[z \longrightarrow \partial_{z_j}(\zeta_k(y,z))\varphi(y+z)\right]\right\}\right) = \\ &\lim_{k\to\infty}\left(u\left\{y \longrightarrow \partial_j v\left[z \longrightarrow \zeta_k(y,z)\varphi(y+z)\right]\right\} + u\left\{y \longrightarrow v\left[z \longrightarrow \partial_{z_j}(\zeta_k(y,z))\varphi(y+z)\right]\right\}\right). \end{aligned} \tag{4.20}$$

Let us observe that $\partial_{z_j}(\zeta_k(y,z)) = \left(\partial_{z_j}(\zeta_k(y,z)) + \zeta_k\right) - \zeta_k$, and since $\zeta_k \xrightarrow{(*)} 1$, we have that $\left(\partial_{z_j}(\zeta_k(y,z)) + \zeta_k\right) \xrightarrow{(*)} 1$. Then we can rewrite the right term on the right-hand side of (4.20):

$$\begin{aligned} u\left\{y \longrightarrow v\left[z \longrightarrow \partial_{z_j}(\zeta_k(y,z))\varphi(y+z)\right]\right\} = (u \times v)\left[(y,z) \to \partial_{z_j}(\zeta_k(y,z))\varphi(y+z)\right] = \\ (u \times v)\left[(y,z) \to \left(\partial_{z_j}(\zeta_k(y,z)) + \zeta_k\right)\right] - (u \times v)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] \end{aligned}$$

The left term tends to $(u * v)$, but the right term converges also to $(u * v)$, so this whole term tends to zero. So we have that

$$\begin{aligned} &\partial_j(u * v)(\varphi) = \\ &\lim_{k\to\infty}\left(u\left\{y \longrightarrow \partial_j v\left[z \longrightarrow \zeta_k(y,z)\varphi(y+z)\right]\right\}\right) = \\ &\lim_{k\to\infty}\left(u \times \partial_j v\right)\left[(y,z) \to \zeta_k(y,z)\varphi(y+z)\right] = (u * \partial_j v)(\varphi), \end{aligned}$$

which gives our statement. □

Corollary 4.29. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, then for all α multi-indexes*

$$\partial^\alpha u = \partial^\alpha(u * \delta) = u * \partial^\alpha \delta_0.$$

Chapter 5

Fundamental solutions

Our goal here is to define fundamental solutions, and show that the solution of a PDE is in the form $u_f * F$, where u_f is the fundamental solution and F is the right-hand side of the equation.

First, to make the formulas more simple, we define the following notation.

Definition 5.1. Let P be an n-variable polynomial (with variables $\xi = (\xi_1, \xi_2, \dots \xi_n)$) given in the form

$$P(\xi) = \sum_{|\alpha| \le k} a_\alpha \xi^\alpha,$$

in which $a_\alpha \in \mathbb{R}$, and $\xi^\alpha = (\xi_1^{\alpha_1}, \xi_2^{\alpha_2}, \dots \xi_n^{\alpha_n})$.
Then the **linear differential operator with constant coefficients** $P(\partial) : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ is defined as

$$P(\partial)u := \sum_{|\alpha| \le k} a_\alpha \partial^\alpha u.$$

Using this notation we can rewrite a linear partial differential equation with constant coefficients in the form

$$P(\partial)u = F, \tag{5.1}$$

where $F \in \mathcal{D}'(\mathbb{R}^n)$ is a given distribution, and we search for $u \in \mathcal{D}'(\mathbb{R}^n)$.
(All the previous definitions also hold if $a_\alpha \in C^\infty(\mathbb{R}^n)$.)

Now we define the fundamental solutions:

Definition 5.2. Let $E \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$P(\partial)E = \delta_0, \qquad \text{in } \mathbb{R}^n.$$

Then we call E **the fundamental solution of the equation** (5.1).

With this, we can now state and prove the main result of this section, which was mentioned many times before.

Theorem 5.1. *Let E be the fundamental solution of* (5.1)*. If $E * F$ exists for $F \in \mathcal{D}'(\mathbb{R}^n)$, then*

$$P(\partial)(E * F) = F, \qquad \text{in } \mathbb{R}^n. \tag{5.2}$$

Also, equation (5.1) *has at most one such solution for which $u * E$ exists.*

Proof. First we prove the first part of the theorem, and then the second part.

1. **First part:** If $E * F$ exists, then by writing it inside (5.1), we get:

$$P(\partial)(E * F) = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha (E * F) =$$

Now we apply the property of differentiation (Proposition 4.28):

$$= \sum_{|\alpha| \leq k} a_\alpha \left[(\partial^\alpha E) * F \right] =$$

Then by linearity (Proposition 4.26):

$$= \left(\sum_{|\alpha| \leq k} a_\alpha \, \partial^\alpha E \right) * F = (P(\partial)E) * F =$$

Since E is a fundamental solution (and by Proposition 4.24)

$$= \delta_0 * F = F,$$

so we get the statement.

2. **Second part:** Let us suppose that we have two solutions $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$, i.e. $P(\partial)u_j = F$ for $j = 1, 2$. Now we prove the uniqueness by a standard technique: we define a new distribution $u := u_1 - u_2$, and then we show that it is zero.
By the linearity of operator $P(\partial)$, we get $P(\partial)u = 0$. Also,

$$u * E = u_1 * E - u_2 * E,$$

and this convolution exists. Then, since E is a fundamental solution, and by Propositions 4.24, 4.26 and 4.28:

$$u = u * \delta_0 = u * (P(\partial)E) = u * \left(\sum_{|\alpha| \leq k} a_\alpha \, \partial^\alpha E \right) = \sum_{|\alpha| \leq k} a_\alpha \, (u * \partial^\alpha E) =$$

$$= \sum_{|\alpha| \leq k} a_\alpha \, \partial^\alpha (u * E) = \sum_{|\alpha| \leq k} a_\alpha \, (\partial^\alpha u * E) = \left(\sum_{|\alpha| \leq k} a_\alpha \, \partial^\alpha u \right) * E = 0 * E = 0,$$

so we get that $u_1 = u_2$.

This completes the proof. □

So basically (5.2) means that $E * F$ is the solution of equation (5.1). This theorem is pretty useful, since if we can calculate a fundamental solution of an equation, then we only have to calculate the convolution with the right-hand side, and we get the solution. This seems to be easy, but of course in real life applications finding a fundamental solution can be hard.

In the next section we list a few examples of fundamental solutions (you do not have to memorize them for the exam).

Examples

1. **Wave equation:**

   (a) *One dimension:*

   Proposition 5.2. *Let $E \in L^1_{loc}(\mathbb{R}^2)$, such that*

   $$E(t,x) := \frac{1}{2}H(t-|x|),$$

   in which H is the Heaviside function. Then E is a fundamental solution of the one dimensional wave equation in the distributional sense, i.e. the following holds on $\mathbb{R}^2$:

   $$\partial_t^2 T_E - \partial_x^2 T_E = \delta_0.$$

   (b) *Two dimensions:* Let

   $$E(t,x) = \frac{H(t-|x|)}{2\pi\sqrt{t^2-|x|^2}},$$

   $x = (x_1, x_2) \in \mathbb{R}^2$ and H is the Heaviside function.

   Proposition 5.3. *Then the following holds in $\mathbb{R}^3$:*

   $$\partial_t^2 T_E - \partial_{x_1}^2 T_E - \partial_{x_2}^2 T_E = \delta_0.$$

   (c) *Three dimensions:* Here E is not a regular distribution, but is defined as:

   $$E(\varphi) := \int_0^\infty \left(\frac{1}{4\pi t} \int_{S(0,t)} \varphi(x) d\sigma_x \right) dt,$$

   in which $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $S(0,t)$ is the surface of the ball centered at 0 with radius t, and σ_x is the surface measure.

   Proposition 5.4. *Then $E \in \mathcal{D}'(\mathbb{R}^3)$ and*

   $$\partial_t^2 E - \partial_{x_1}^2 E - \partial_{x_2}^2 E - \partial_{x_3}^2 E = \delta_0.$$

   Remark 5.1. The support of E here is part of a convex cone (for a given t, there is a convex set at which the solution is non-zero), meaning that the wave propagates only with a finite speed. This is also true for the solutions of hyperbolic equations in the canonical form.

2. **Heat equation:** Let us define $E(t,x)$ as

   $$E(t,x) := \begin{cases} \dfrac{1}{(2\sqrt{\pi t})^n} \exp\left(-\dfrac{|x|^2}{4t}\right) & , \quad \text{if } t > 0, x \in \mathbb{R}^n \\ 0 & , \quad \text{if } t \le 0, x \in \mathbb{R}^n \end{cases}$$

   Proposition 5.5. *Here $E \in L^1_{loc}(\mathbb{R}^{n+1})$ and*

   $$\partial_t T_E - \sum_{j=1}^n \partial_{x_j}^2 T_E = \delta_0 \qquad \text{in } \mathbb{R}^{n+1}.$$

For the proof, see Exercise 4 on Practice 6.

Remark 5.2. The support of E here is part of a half space (for a given t, the function is non-zero at almost every point of $\mathbb{R}_+ \times \mathbb{R}^n$), meaning that the heat propagates with infinite speed (since if the initial heat-distribution had compact support, after any small time there will be points arbitrary far from the initial distribution at which the solution is non-zero) - see Remark 7.4. (In the previous decades some researchers proposed other operators instead of the Laplacian - these are called fractional Laplacian operators, and using them can result in finite propagation of heat.)

3. **Poisson equation:** Let us define $E(x)$ as:

$$E(x) := \begin{cases} -\dfrac{1}{(n-2)\omega_n}\dfrac{1}{|x|^{n-2}} & , \quad \text{if } n \geq 3, x \in \mathbb{R}^n \setminus \{0\} \\ -\dfrac{1}{2\pi}\log\left(\dfrac{1}{|x|}\right) & , \quad \text{if } n = 2, x \in \mathbb{R}^n \setminus \{0\}, \end{cases}$$

where ω_n is the area of the surface of the n-dimensional ball of unity.

Proposition 5.6. *Here* $E \in L^1_{loc}(\mathbb{R}^n)$, *and*

$$\Delta T_E = \delta_0 \qquad \text{on } \mathbb{R}^n \setminus \{0\}.$$

Remark 5.3. It is worth mentioning here that the main method of finding the fundamental solution of an equation involves the Fourier transform, which is also an important tool in the theory of partial differential equations.
Also, the Malgrange-Ehrenpreis theorem states that every linear partial differential equation with constant coefficients has a fundamental solution, but these topics are far beyond the scope of this course.

In the next chapters we will observe some concrete PDEs and get their solutions using the fundamental solutions.

Chapter 6

Cauchy problem of the wave equation

As it was mentioned at the end of the previous chapter, in the next chapters we observe different concrete types of equations, and try to apply the theory of fundamental solutions to them to acquire the solutions of these problems. This week we consider the wave equation, and next week we will talk about the heat equation, and later the boundary-value problems will be discussed.

As it was defined in Section 1.2.2, the wave equation has the form

$$\partial_t^2 u - \Delta u + cu = f, \tag{6.1}$$

in which $c \in \mathbb{R}$, and Δ is the Laplacian operator taken in the space variables (and not in t), and f is a given (usually sufficiently smooth) function (the "source term").

Definition 6.1. The **classical solution** of equation (6.1) is such a function $u \in C^2(\mathbb{R}_+^{n+1})$ for which (6.1) holds ($t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $\mathbb{R}_+^{n+1} = \mathbb{R}^+ \times \mathbb{R}^n$), u, $\partial_t u \in C(\overline{\mathbb{R}_+^{n+1}})$, and also

$$\begin{aligned} u(0,x) &= g(x) \qquad (x \in \mathbb{R}^n) \\ \partial_t u(0,x) &= h(x) \qquad (x \in \mathbb{R}^n) \end{aligned}$$

in which $g, h : \mathbb{R}^n \to \mathbb{R}$ given functions.

So here g and h are the initial conditions, and since we defined our problem on the whole $\mathbb{R}^n$ (meaning that $x \in \mathbb{R}^n$), we do not need any boundary conditions ("the domain does not have a boundary"). One can say that the "classical solution" is the intuitive one (to be able to solve the equation, we of course need u to be twice differentiable). However, using the theory of distributions, we will define a weak solution also. Since this will be a generalization of the classical solution, we will mostly talk about the general one, and only consider the classical one as a special case of the weak one.

6.1 Generalized (or weak) solutions

Our first goal is to be able to define our problem not only on $\mathbb{R}_+^{n+1}$, but also on $\mathbb{R}^{n+1}$ (so t can also be negative). For this, we have to generalize u and f:

$$\widetilde{u}(t,x) := \begin{cases} u(t,x) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

$$\widetilde{f}(t,x) := \begin{cases} f(t,x) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Here $\widetilde{u} \in L^1_{loc}(\mathbb{R}^{n+1})$ (since $u \in C(\overline{\mathbb{R}^{n+1}_+})$), and we also have to suppose that $\widetilde{f} \in L^1_{loc}(\mathbb{R}^{n+1})$ - for this, $f \in C(\overline{\mathbb{R}^{n+1}})$ is enough.

Now we state the key proposition of this section, which talks about the generalization of the classical problem.

Proposition 6.1. *Suppose that u is a classical solution of equation (6.1), and $\widetilde{f} \in L^1_{loc}(\mathbb{R}^{n+1})$. Then for $\widetilde{u}$ (defined as before), the following equation holds in $\mathbb{R}^{n+1}$.*

$$\partial_t^2 T_{\widetilde{u}} - \Delta T_{\widetilde{u}} + c T_{\widetilde{u}} = T_{\widetilde{f}} + \delta' \times T_g + \delta \times T_h. \tag{6.2}$$

Proof. The statement is proved by direct calculations.

Let $\varphi \in \mathcal{D}(\mathbb{R}^{n+1})$, then

$$(\partial_t^2 T_{\widetilde{u}} - \Delta T_{\widetilde{u}} + c T_{\widetilde{u}})(\varphi) = (\partial_t^2 T_{\widetilde{u}})(\varphi) - (\Delta T_{\widetilde{u}})(\varphi) + c(T_{\widetilde{u}})(\varphi) =$$

By the definition of derivative of distribution:

$$= (T_{\widetilde{u}})(\partial_t^2 \varphi) - (T_{\widetilde{u}})(\Delta\varphi) + c(T_{\widetilde{u}})(\varphi) = \int_{\mathbb{R}^{n+1}} \widetilde{u} \left(\partial_t^2 \varphi - \Delta\varphi + c\varphi\right) =$$

$$= \lim_{\varepsilon \to 0+} \int_\varepsilon^\infty \int_{\mathbb{R}^n} u \left(\partial_t^2 \varphi - \Delta\varphi + c\varphi\right) \tag{6.3}$$

Here the first integral corresponds to variable t (time) and the others to the space ones.

First let us observe the first term of (6.3). For this, we will change the order the integrals, and first only consider the inner one, i.e. the one in t:

$$\int_\varepsilon^\infty u(t,x)\partial_t^2 \varphi(t,x)dt =$$

Now by partial integration, and using the fact that φ has compact support (it is zero at infinity) we get that

$$= -u(\varepsilon,x)\partial_t\varphi(\varepsilon,x) - \int_\varepsilon^\infty \partial_t u(t,x)\partial_t\varphi(t,x)dt =$$

By the same arguments:

$$= -u(\varepsilon,x)\partial_t\varphi(\varepsilon,x) + \partial_t u(\varepsilon,x)\varphi(\varepsilon,x) + \int_\varepsilon^\infty \partial_t^2 u(t,x)\varphi(t,x)dt$$

Also, by the second Green formula (which will be proved in Chapter 8) we know that

$$\int v\Delta u - u\Delta v = \int (v\partial_\nu u - u\partial_\nu v)\partial\sigma$$

in which ∂_ν is the normal derivative. Then by applying this to our second term of (6.3) (without the integral in time) we get:

$$\int_{\mathbb{R}^n} u\Delta\varphi = \int_{\mathbb{R}^n} (\Delta u)\varphi$$

in which we used that φ has compact support (so its values on the boundary and its normal derivatives are zero).

Using these two observations we get that

$$\int_{\varepsilon}^{\infty}\int_{\mathbb{R}^n} u\left(\partial_t^2\varphi - \Delta\varphi + c\varphi\right) =$$

$$= -\int_{\mathbb{R}^n} u(\varepsilon,x)\partial_t\varphi(\varepsilon,x)dx + \int_{\mathbb{R}^n}\partial_t u(\varepsilon,x)\varphi(\varepsilon,x)dx +$$
$$+\int_{\varepsilon}^{\infty}\int_{\mathbb{R}^n}\left(\partial_t^2 u(t,x) - \Delta u + cu\right)\varphi(t,x)dtdx$$

Note that in the last term, we have $\partial_t^2 u(t,x) - \Delta u + cu$, which equals to f on $\mathbb{R}_+^{n+1}$ because of our equation.

Then if we take the limit of the expression above, we can use Lebesgue's theorem (since φ has compact support), and we get

$$\lim_{\varepsilon\to 0+}\int_{\varepsilon}^{\infty}\int_{\mathbb{R}^n} u\left(\partial_t^2\varphi - \Delta\varphi + c\varphi\right) =$$

$$= -\int_{\mathbb{R}^n} u(0,x)\partial_t\varphi(0,x)dx + \int_{\mathbb{R}^n}\partial_t u(0,x)\varphi(0,x)dx + \int_{\mathbb{R}_+^{n+1}} f(t,x)\varphi(t,x)dtdx =$$

Now we use the initial conditions:

$$= -\int_{\mathbb{R}^n} g(x)\partial_t\varphi(0,x)dx + \int_{\mathbb{R}^n} h(x)\varphi(0,x)dx + \int_{\mathbb{R}_+^{n+1}} f(t,x)\varphi(t,x)dtdx \quad (6.4)$$

The only thing left to show is that (6.4) is indeed the right-hand side of (6.2).

For the first term of (6.4):

$$-\int_{\mathbb{R}^n} g(x)\partial_t\varphi(0,x)dx = -\left(\delta_0 \times T_g\right)\left(\partial_t\varphi\right) = \partial_t\left(\delta_0 \times T_g\right)(\varphi) = \left(\delta_0' \times T_g\right)(\varphi)$$

Also, for the second term of (6.4):

$$\int_{\mathbb{R}^n} h(x)\varphi(0,x)dx = T_h\left\{x \to \varphi(0,x)\right\} = T_h\left\{x \to \delta_0[t \to \varphi(t,x)]\right\} =$$

Now by applying the commutative property of the Cartesian product of distributions:

$$= (T_h \times \delta_0)(\varphi) = (\delta_0 \times T_h)(\varphi)$$

Also, the third term of (6.4) is clearly $T_f(\varphi)$, which is by definition the same as $T_{\widetilde{f}}(\varphi)$. Consequently,

$$\left(\partial_t^2 T_{\widetilde{u}} - \Delta T_{\widetilde{u}} + cT_{\widetilde{u}}\right)(\varphi) = (\delta' \times T_g)(\varphi) + (\delta \times T_h)(\varphi) + (T_{\widetilde{f}})(\varphi),$$

which gives our statement. □

Remark 6.1. It is important to note that $\operatorname{supp}\left(T_f + \delta' \times T_g + \delta \times T_h\right) \subset \overline{\mathbb{R}_+^{n+1}}$, which means that the next definition makes sense.

Definition 6.2. Let $F \in \mathcal{D}'(\mathbb{R}^{n+1})$ be a given functional in a way that $\operatorname{supp}(F) \subset \overline{\mathbb{R}_+^{n+1}}$. Then the **general (or weak) solution** of (6.1) is such a distribution $v \in \mathcal{D}'(\mathbb{R}^{n+1})$, for which

$$\partial_t^2 v - \Delta v + cv = F \qquad (\text{in } \mathbb{R}^{n+1}) \quad (6.5)$$

and $\operatorname{supp}(v) \subset \overline{\mathbb{R}_+^{n+1}}$. Equation (6.5) is called the *general Cauchy-problem of the wave equation.*

Then by using Proposition 6.1 we can state the following result, which is the direct consequence of the definition and Proposition 6.1.

Proposition 6.2. *If u is a classical solution of* (6.1) *and $\widetilde{f} \in L^1_{loc}(\mathbb{R}^{n+1})$, then the distribution $v := T_{\widetilde{u}}$ (in which $\widetilde{u}$ is the same as defined before) is a solution of the general equation* (6.5) *with $F = T_{\widetilde{f}} + \delta' \times T_g + \delta \times T_h$.*

So Proposition 6.2 states that the generalized solution is indeed a generalization of the classical solution.

Now we state the main result of this section, which talks about the existence and uniqueness of solutions of (6.5).

Theorem 6.3. *The generalized Cauchy-problem* (6.5) *has a unique solution $v \in \mathcal{D}'(\mathbb{R}^{n+1})$ in the form $v = E * F$, in which $E \in \mathcal{D}'(\mathbb{R}^{n+1})$ is such a fundamental solution of* (6.5) *for which* $\operatorname{supp}(E) \subset \{(t,x) \in \mathbb{R}^{n+1} : t \geq |x|\}$.

Proof. (Only main ideas)

1. **Step 1:** It can be shown that there exists such a fundamental solution of (6.5) denoted by $E \in \mathcal{D}'(\mathbb{R}^{n+1})$ for which $\operatorname{supp}(E) \subset \{(t,x) \in \mathbb{R}^{n+1} : t \geq |x|\}$.

   Then it can be shown that $E * F$ exists, and by Theorem 5.1 we know that for $v = E * F$,

   $$\partial_t^2 v - \Delta v + cv = F \qquad (\text{in } \mathbb{R}^{n+1}).$$

   Moreover,

   $$\operatorname{supp}(v) = \operatorname{supp}(E * F) \subset \overline{\operatorname{supp}(E) + \operatorname{supp}(F)} \subset \overline{\overline{\mathbb{R}^{n+1}_+} + \overline{\mathbb{R}^{n+1}_+}} = \overline{\mathbb{R}^{n+1}_+}.$$

2. **Step 2:** By Theorem 5.1, there is at most one $v \in \mathcal{D}'(\mathbb{R}^{n+1})$ for which $v * E$ exists, and since $\operatorname{supp}(E) \subset \{(t,x) \in \mathbb{R}^{n+1} : t \geq |x|\}$, then there exists at most one $v \in \mathcal{D}'(\mathbb{R}^{n+1})$ for which $\operatorname{supp}(v) \subset \overline{\mathbb{R}^{n+1}_+}$ (since $v * E$ exists for such v distributions).

By Step 1, the distribution defined in Step 2 is a solution, so we proved the statement. □

Now we can get the following result from Theorem 6.3.

Corollary 6.4. *The classical problem* (6.1) *has at most one classical solution.*

Proof. Proceeding towards contradiction, let us suppose that there are two solutions, namely u_1 and u_2. Then the function $u := u_1 - u_2$ is a solution of the equation

$$\partial_t^2 u - \Delta u + cu = 0 \qquad (\text{in } \mathbb{R}^{n+1}),$$

and also $u(0,x) = 0$ and $\partial_t u(0,x) = 0$ for all $x \in \mathbb{R}^n$. Therefore the distribution $v := T_{\widetilde{u}}$ (in which $\widetilde{u}$ is defined as before) is a solution of the equation

$$\partial_t^2 v - \Delta v + cv = 0.$$

We know that $\operatorname{supp}(v) \subset \overline{\mathbb{R}^{n+1}_+}$, and then by Theorem 6.3 the only solution is $v = 0$, so $u = 0$. □

6.2 Classical solutions

At the end of the previous section, we showed that the classical problem has at most one solution. The next theorem talks about the form of such solutions, if they exist.

Theorem 6.5 (The formula of D'Alembert)**.** *Let $n = 1$, $c = 0$ and suppose that $f \in C^1(\overline{\mathbb{R}^2_+})$, $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then equation (6.1) has a unique solution $u \in C^2(\overline{\mathbb{R}^2_+})$, and it has the form*

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau,\xi)d\xi d\tau + \frac{1}{2}\left(g(x+t)+g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi \tag{6.6}$$

Proof. The uniqueness comes from Corollary 6.4. We prove the existence in three steps: first we show that the function (6.6) is a weak solution in the distribution sense of equation (6.5). Then, in step 2 we show that if the given functions are smooth enough, then $u \in C^2(\overline{\mathbb{R}^2_+})$, and then in step 3 we show that this function satisfies the initial conditions.

1. **Step 1:** We know that the unique solution is in the form $E * F$, in which

$$E(t,x) = H(t-|x|)/2,$$

by Proposition 5.2, and also

$$F = T_{\widetilde{f}} + \delta' \times T_g + \delta \times T_h.$$

Now we prove that the convolution $E*F$ also exists in the classical (or function) sense.

$$(E * \widetilde{f})(t,x) = \int_{\mathbb{R}^2} \widetilde{f}(\tau,\xi)E(t-\tau,x-\xi)d\tau d\xi =$$

$$= \int_{|x-\xi|\le t-\tau}\int_0^t \frac{1}{2}f(\tau,\xi)d\tau d\xi = \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} f(\tau,\xi)d\xi d\tau.$$

In the proof of Proposition 6.1 we showed that

$$(\delta \times T_h)(\varphi) = \int_{\mathbb{R}} h(x)\varphi(0,x)dx,$$

which means that

$$(E*(\delta\times T_h))(\varphi) = (E*(\delta\times T_h))\left[(t,x,\tau,\xi)\to\psi(\tau,\xi)\chi(t,x)\varphi(t+\tau,x+\xi)\right] =$$

$$= (\delta\times T_h)\left\{(\tau,\xi)\to\psi(\tau,\xi)T_E\left[(t,x)\to\chi(t,x)\varphi(t+\tau,x+\xi)\right]\right\} =$$

$$= \int_{\mathbb{R}} h(\xi)\psi(0,\xi)\int_{\mathbb{R}^2} E(t,x)\chi(t,x)\varphi(t,x+\xi)dtdxd\xi,$$

in which $\psi = 1$ in a neighborhood of the half-space $\mathbb{R}^2_+$ and $\chi = 1$ in a neighborhood of the cone $\{(t,x)\in\mathbb{R}^2 : t \ge |x|\}$. Then by applying the theorem of Fubini,

$$(E*(\delta\times T_h))(\varphi) = \int_{\mathbb{R}} h(\xi)\psi(0,\xi)\int_{\mathbb{R}^2} E(t,x)\chi(t,x)\varphi(t,x+\xi)dtdxd\xi =$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}} h(\xi)\psi(0,\xi)E(t,x)\chi(t,x)\varphi(t,x+\xi)d\xi dtdx =$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}} h(\xi)\psi(0,\xi)E(t,\widetilde{x}-\xi)\chi(t,\widetilde{x}-\xi)\varphi(t,\widetilde{x})d\xi dtd\widetilde{x} =$$

$$= \int_{\mathbb{R}^2}\varphi(t,\widetilde{x})\left(\frac{1}{2}\int_{|\widetilde{x}-\xi|\le t} h(\xi)\psi(0,\xi)\chi(t,\widetilde{x}-\xi)d\xi\right)dtd\widetilde{x} =$$

$$= \int_{\mathbb{R}^2} \varphi(t,\widetilde{x}) \left(\frac{1}{2} \int_{|\widetilde{x}-\xi|\le t} h(\xi) d\xi \right) dt d\widetilde{x}.$$

This basically means that $E * (\delta \times T_h) = T_v$, where

$$v(t,\widetilde{x}) = \frac{1}{2} \int_{|\widetilde{x}-\xi|\le t} h(\xi) d\xi = \frac{1}{2} \int_{\widetilde{x}-t}^{\widetilde{x}+t} h(\xi) d\xi.$$

By similar arguments,

$$(E * (\delta' \times T_g))(\varphi) = -\int_{\mathbb{R}^2} \partial_t \varphi(t,\widetilde{x}) \left(\frac{1}{2} \int_{|\widetilde{x}-\xi|\le t} g(\xi) d\xi \right) dt d\widetilde{x} =$$

$$= -\int_{\mathbb{R}^2} \partial_t \varphi(t,\widetilde{x}) \left(\frac{1}{2} \int_{\widetilde{x}-t}^{\widetilde{x}+t} g(\xi) d\xi \right) dt d\widetilde{x}. \tag{6.7}$$

Now we use that φ has a compact support, and the fact that

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \int_{\widetilde{x}-t}^{\widetilde{x}+t} g(\xi) d\xi \right) = \frac{1}{2} \left(g(x+t) + g(x-t) \right).$$

With these, if we apply integration by parts on (6.7), we get

$$(E * (\delta' \times T_g))(\varphi) = \int_{\mathbb{R}^2} \varphi(t,\widetilde{x}) \left(\frac{1}{2} (g(x+t) + g(x-t)) \right) dt d\widetilde{x},$$

which means that $E * (\delta \times T_h) = T_w$, where

$$w(t,x) = \frac{1}{2} (g(x+t) + g(x-t)).$$

To sum it up, we got that if $F = T_{\widetilde{f}} + \delta' \times T_g + \delta \times T_h$, then $E * F = T_u$, where u has the form (6.6).

2. **Step 2:** In this step we show that the function u defined in (6.6) is in $C^2(\overline{\mathbb{R}^2_+})$. This is a consequence of the smoothness of the functions involved in the formula and the properties of parametric integrals. Since u is twice continuously differentiable, its classical and weak (or distributional) derivatives are the same. Now since u is a solution of the equation in the weak sense, then by Proposition 6.2 u is a classical solution too.

3. **Step 3:** In this step we show that the initial conditions are satisfied. By the formula it is evident that $u(0,x) = h(x)$, and also

$$\partial_t u(t,x) = \frac{1}{2} \int_0^t f(\tau, x+(t-\tau)) + f(t, x-(t-\tau)) d\tau +$$

$$+ \frac{1}{2} (g'(x+t) - g'(x-t)) + \frac{1}{2} (h(x+t) + h(x-t)),$$

meaning that $\partial_t u(0,x) = h(x)$.

Then our statement is proved. □

Formula (6.6) is called **D'Alembert's formula**. (You don't have to know the correct form by heart, but you have to be able to recognize it.)

If $n > 1$, then a similar, but more complicated formula can be proved.

Remark 6.2 (Huygens principle). An interesting phenomena happens when we move into more dimensions: the solutions might have different behavior.

If we drop a little rock into a pond which have a still surface, waves will start to form, and not only one wave, but several different ones, starting from the point of impact. This process is called wave diffusion, and happens for the solutions of the wave equation in every even dimensional space (the surface of the lake is a two-dimensional space).

However, let us think of a lamp. When we switch it on, its light will travel like a wave with the speed of light, but we will only have just one wavefront, and no several other waves. This wavefront behavior happens in the case of odd dimensional spaces (we live in a 3-dimensional space).

This phenomena is called the *Huygens principle*.

In the next chapter we observe the heat equation, and state similar theorems as in this one.

Chapter 7

Cauchy problem of the heat equation

In the previous chapter we focused on the wave equation. After we defined the classical problem, we generalized it, and after proving the existence and uniqueness of solutions for this latter one, we also saw the form of the classical solution in one dimension. This time we proceed in a similar way, but now defining all these things for the heat equation.

As it was defined in Section 1.2.1, the heat equation has the form

$$\partial_t u - \Delta u = f \tag{7.1}$$

in which the Laplace operator Δ is only taken in the 2nd, 3rd, ... nth variable (since the first one is time, and the Laplacian describes movement in space).

Note that since in equation (7.1) we only take the first derivative of u with respect to time, we only need it to be once continuously differentiable in t. Because of this, we define a new set of functions, namely:

$$C^{1,2}(\mathbb{R}_+^{n+1}) := \{u : \mathbb{R}_+^{n+1} \to \mathbb{R} : \partial_t u \in C(\mathbb{R}_+^{n+1}), \partial_{ij} u \in C(\mathbb{R}_+^{n+1}), i, j \neq 0\}.$$

So this space has all the functions which can be continuously differentiated with respect to t, and can be continuously differentiated twice with respect to the space variables.

As in the previous chapter, we first define the classical solutions of equation (7.1).

Definition 7.1. The **classical solution** of equation (7.1) is such a function $u \in C^{1,2}(\mathbb{R}_+^{n+1})$, for which

- (7.1) holds (in the classical sense)
- $u \in C(\overline{\mathbb{R}_+^{n+1}})$
- $u(0, x) = g(x)$ $(x \in \mathbb{R}^n)$ holds, where $g : \mathbb{R}^n \to \mathbb{R}$ is a given function.

As in the case of the wave equation, instead of observing the classical solutions, we define and state results for the generalized solutions first, and then they will also hold for the classical ones.

7.1 Generalized (or weak) solutions

As before, our first goal is to define all of our functions in $\mathbb{R}^{n+1}$ instead of $\mathbb{R}_+^{n+1}$.

$$\widetilde{u}(t, x) := \begin{cases} u(t, x) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

$$\widetilde{f}(t,x) := \begin{cases} f(t,x) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Here $\widetilde{u} \in L^1_{loc}(\mathbb{R}^{n+1})$ (since $u \in C(\overline{\mathbb{R}^{n+1}_+})$), and we also have to suppose that $\widetilde{f} \in L^1_{loc}(\mathbb{R}^{n+1})$ - for this, $f \in C(\overline{\mathbb{R}^{n+1}_+})$ is enough.

Now we state the key result of this chapter.

Proposition 7.1. *Suppose that u is a classical solution of (7.1), and $\widetilde{f} \in L^1_{loc}(\mathbb{R}^{n+1})$. Then the following holds for $\widetilde{u}$:*

$$\partial_t T_{\widetilde{u}} - \Delta T_{\widetilde{u}} = T_{\widetilde{f}} + \delta \times T_g.$$

The proof is similar to the proof of Proposition 6.1, so it is skipped.

Remark 7.1. Here $\operatorname{supp}(T_{\widetilde{f}} + \delta \times T_g) \subset \overline{\mathbb{R}^{n+1}_+}$, so the next definition makes sense.

Definition 7.2. Let $F \in \mathcal{D}'(\mathbb{R}^{n+1})$ be a given functional in a way that $\operatorname{supp}(F) \subset \overline{\mathbb{R}^{n+1}_+}$. Then the **general (or weak) solution** of (7.1) is such a distribution $v \in \mathcal{D}'(\mathbb{R}^{n+1})$ for which

$$\partial_t v - \Delta v = F \qquad (\text{in } \mathbb{R}^{n+1}) \tag{7.2}$$

and $\operatorname{supp}(v) \subset \overline{\mathbb{R}^{n+1}_+}$. Equation (7.2) is called the **general Cauchy-problem of the heat equation**.

Then by using Proposition 7.1 we can state the following result, which is the direct consequence of the definition and Proposition 7.1.

Proposition 7.2. *If u is a classical solution of (7.1) and $\widetilde{f} \in L^1_{loc}(\mathbb{R}^{n+1})$, then the distribution $v := T_{\widetilde{u}}$ (in which $\widetilde{u}$ is the same as defined before) is a solution of the general equation (7.2) with $F = T_{\widetilde{f}} + \delta \times T_g$.*

So Proposition 7.2 states that the generalized solution is indeed a generalization of the classical solution.

Now we would like to compute such a generalized solution. We know that these can be written as $v = E * F$, in which F is the right-hand side of(7.2), and E is the fundamental solution of the equation, namely:

$$E(t,x) := \begin{cases} \dfrac{1}{(2\sqrt{\pi t})^n} \exp\left(\dfrac{-|x|^2}{4t}\right) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

The problem here is that the convolution $E * F$ might not exist for all F functions: we need that the functions $F \exp(-|x|^2)$ are integrable. However, one can define such a set of distribution $\widetilde{\mathcal{M}}$ for which $E * F$ exists.

We are introducing a special set of functions, for which this property will hold.

Definition 7.3. Let $\mathcal{M}$ be the set of such measureable functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$ for which

- $f(t,x) = 0$ if $t < 0$, and
- $f|_{[0,T]\times\mathbb{R}^n} \in L^\infty([0,T] \times \mathbb{R}^n)$ for all $T > 0$.

So $\mathcal{M}$ contains such functions which are zero for $t < 0$ (before our process), and which are bounded (in the L^∞-sense) on any finite time interval. The next proposition states that these functions are good choices in the sense that the convolution $E * f$ is well defined for them.

Proposition 7.3. *If $f \in \mathcal{M}$, then $E * f$ exists, and $E * f \in \mathcal{M}$.*

Proof. It is easy to see that if $f \in \mathcal{M}$, then $f \in L^1_{loc}(\mathbb{R}^{n+1})$. It can also be proved that $E \in L^1_{loc}(\mathbb{R}^{n+1})$ (see Exercise 4 from Practice 6, which is also copied here).

Let $K \subset \mathbb{R} \times \mathbb{R}^n$ be a compact set. Then there is a $T > 0$, for which $K \subset [-T, T] \times \mathbb{R}^n$. Let $t > 0$ be fixed, and we apply the substitution $\xi = \frac{x}{2\sqrt{t}}$. Then the absolute value of the Jacobian of this transformation is $\left(2\sqrt{t}\right)^n$, meaning that

$$\int_{\mathbb{R}^n} E(t,x)dx = \int_{\mathbb{R}^n} \frac{1}{\left(2\sqrt{\pi t}\right)^n} e^{-|\xi|^2} \left(2\sqrt{t}\right)^n d\xi = \frac{1}{\left(\sqrt{\pi}\right)^n} \int_{\mathbb{R}^n} e^{-|\xi|^2} d\xi = 1.$$

Then by using these arguments, we get

$$\int_K |E| = \int_K E \leq \int_{-T}^{T} \left(\int_{\mathbb{R}^n} E(t,x)dx \right) dt = 2T < \infty.$$

Now by definition:

$$(E * f)(t,x) = \int_{\mathbb{R}^{n+1}} E(\tau,\xi) f(t-\tau, x-\xi) d\tau d\xi \leq$$

Since if $t < 0$ we know that $f(t,x) = 0$, so

$$\leq \int_0^t \int_{\mathbb{R}^n} E(\tau,\xi) |f(t-\tau, x-\xi)| d\xi d\tau \leq$$

$$\leq \|f\|_{L^\infty([0,t]\times\mathbb{R}^n)} \int_0^t \int_{\mathbb{R}^n} E(\tau,\xi) d\xi d\tau$$

We also know by Exercise 4 (see the small text above) that $\int_{\mathbb{R}^n} E(\tau,\xi)d\xi = 1$, and $E(t,x) = 0$ for $t < 0$, so we got that

$$|(E * f)(t,x)| \leq t \cdot \|f\|_{L^\infty([0,t]\times\mathbb{R}^n)}.$$

Now $t \cdot \|f\|_{L^\infty([0,t]\times\mathbb{R}^n)}$ is finite, so for $T > 0$ we got that $(E * f)|_{(0,T)\times\mathbb{R}^n} \in L^\infty([0,T] \times \mathbb{R}^n)$, so $E * f$ exists (in the function sense), and for $t < 0$ we know that $(E * f)(t,x) = 0$. Consequently, $E * f \in \mathcal{M}$. □

Note that in Proposition 7.3 we are considering functions, but for the generalized solution, we need the convolution of distributions. However, if $E * f$ exists, then we know that $E * f \in L^1_{loc}(\mathbb{R}^{n+1})$, meaning that we can define regular distributions corresponding to them.

Definition 7.4. Let $\widetilde{\mathcal{M}}$ be the set of the distributions in the form

$$F = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha T_{f_\alpha},$$

in which $a_\alpha \in \mathbb{R}$, $f_\alpha \in \mathcal{M}$ and $m \in \mathbb{N}$ are arbitrary.

So now we have a set of distributions - the next proposition states that they will be an appropriate choice.

Proposition 7.4. *If $F \in \widetilde{\mathcal{M}}$, then $E * F$ exists in the distribution sense, and $E * F \in \widetilde{\mathcal{M}}$.*

Proof. The statement can be proved by direct calculations.

Let us suppose that $F \in \widetilde{\mathcal{M}}$. If $f_\alpha \in \mathcal{M}$, then $E * f_\alpha$ exists in the function sense (because of the previous proposition), and $E * f_\alpha \in \mathcal{M}$. Then by the theory of the convolution of distributions,

$$T_E * T_{f_\alpha} = T_{E * f_\alpha},$$
$$\partial^\alpha \left(T_E * T_{f_\alpha} \right) = T_E * \partial^\alpha T_{f_\alpha}.$$

From which we get that

$$E * F = T_E * \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha T_{f_\alpha} = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha (T_E * T_{f_\alpha}) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha T_{E * f_\alpha}$$

and then since $E * f_\alpha \in \mathcal{M}$, we get that $E * F \in \widetilde{\mathcal{M}}$. □

This means that for $F \in \widetilde{\mathcal{M}}$ the convolution $E * F$ exists, so we can state our main theorem.

Theorem 7.5. *Let $F \in \widetilde{\mathcal{M}}$. Then the general parabolic Cauchy-problem (7.2) has a unique solution $v \in \widetilde{\mathcal{M}}$, i.e. $v = E * F$, in which E is the fundamental solution defined above and F is the right-hand side of the equation.*

Proof. Since $F \in \widetilde{\mathcal{M}}$, then by Proposition 7.4 $E * F$ exists and it is in $\widetilde{\mathcal{M}}$. By Theorem 5.1 we know that the solution is in the form $v = E * F$. Moreover, by Proposition 4.27,

$$\operatorname{supp}(v) \subset \overline{\operatorname{supp}E + \operatorname{supp}F} \subset \overline{\mathbb{R}^{n+1}_+ + \mathbb{R}^{n+1}_+} = \overline{\mathbb{R}^{n+1}_+}.$$

Uniqueness is a consequence of Theorem 5.1, since we know that equation (7.2) can at most one such $v \in \mathcal{D}'(\mathbb{R}^{n+1})$ solution for which $v * E$ exists, but by Proposition 7.4 we know that if $v \in \mathcal{M}$, then $v * E$ exists. □

By the previous Propositions we get the following result from the Theorem.

Corollary 7.6. *The classical Cauchy problem has at most one such solution u for which for every $T > 0$ it holds that*

$$u|_{[0,T]\times\mathbb{R}^n} \in L^\infty([0,T] \times \mathbb{R}^n).$$

Remark 7.2. If we do not require the boundedness of the solutions, it might happen that our heat equation has several other solutions. For further details, see Remarks 7.5 and 7.6.

7.2 Classical solutions

By the end of the previous section, we showed that the classical problem has at most one such solution which is bounded. The next theorem (which is stated without a proof) talks about the form of such bounded (and thus, physically important) solutions.

Theorem 7.7. *Suppose that $f \in C^{1,2}(\overline{\mathbb{R}^{n+1}_+})$, $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\partial^k \partial^\alpha f \in L^\infty([0,T] \times \mathbb{R}^n)$, for all $T > 0$, $2k + |\alpha| \le 2$ ($k \ge 0$). Then equation (7.1) has exactly one such solution u which is bounded on all regions $[0,T] \times \mathbb{R}^n$ ($T > 0$), and it has the form*

$$\begin{aligned} u(t,x) = \int_0^t \frac{1}{2\sqrt{\pi(t-\tau)}} \int_{\mathbb{R}^n} f(\tau,\xi) \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right) d\xi d\tau + \\ + \frac{1}{(2\sqrt{\pi t})^n} \int_{\mathbb{R}^n} g(\xi) \exp\left(-\frac{|x-\xi|^2}{4t}\right) d\xi \end{aligned} \tag{7.3}$$

(You are not required to know this formula by heart, but you have to be able to recognize it.)

Proof. Uniqueness is a guaranteed by Corollary 7.6.

For the existence, it can be proved that the solution of the weak problem (7.2) is the regular distribution corresponding to the function (7.3). By the assumptions of the theorem, this functions is locally integrable, and it is a solution of the classical Cauchy problem (7.1). □

Remark 7.3. Note that the smoothness assumption for function g are only needed for the initial condition to hold. It can also be seen that if we only assume that $g \in L^\infty$, then the second term of (7.3) can be differentiated any number of times, which means that even if the initial function g is not continuous, then the solution will be continuous: this is sometimes called the *smoothing effect of the heat equation.* However, if g is continuous, then

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = g(x_0)$$

also holds.

Remark 7.4. If $f = 0$, and g is non-negative, but is zero outside of a compact set, then for any $t > 0$, the solution is positive at any point x. This practically means that if the initial heat was concentrated at just a bounded interval, then it can get to any point in any small time, meaning that the heat transfers infinitely fast. This seems to contradict the principle of Einstein about that nothing can move faster than lightspeed. However, those positive values very far from the initial concentration are very small, so they might not even make sense physically - see Planck temperature.

Remark 7.5. Note that even in the case $f = 0$, $g = 0$, there are infinitely many solutions of the parabolic equation, which also tend to infinity pretty quickly. A construction of such functions were given by Tikhonov. The previous theorem says that there is only one of these solutions which behaves as expected (in a physically reasonable way) and it has the form (7.3).

Remark 7.6. The solution of problem (7.1) with a physically reasonable behavior is also unique on a more general set of functions.
Let us define $\mathcal{M}_\sigma$ to be a set of functions $u : \overline{\mathbb{R}^{n+1}_+} \to \mathbb{R}$, for which for all $T > 0$ there are some c_T, a_T constants for which

$$|u(t,x)| \le c_T \exp\left(a_T |x|^\sigma\right)$$

if $0 \le t \le T$, $x \in \mathbb{R}^n$. Then it can be shown that the classical solution is unique on $\mathcal{M}_2$, but there are infinitely many solutions on $\mathcal{M}_\sigma$ if $\sigma > 2$. However, Widder's theorem states that only one of these infinitely many solutions is non-negative. However, these topics are far beyond the scope of this course.

7.3 Maximum principle and uniqueness of the classical solutions (skipped in 2024)

This whole section was skipped in 2024.

In this section we show another proof for the uniqueness of solutions using a result which is one of the most famous theorems in the theory of PDEs called maximum principle.

Lt us assume that $\Omega \subset \mathbb{R}^n$ and $T > 0$. We are also going to define the parabolic cylinder as $Q_T = (0,T] \times \Omega$ and the parabolic boundary $\Gamma_T = \overline{Q_T} \setminus Q_T = (\{0\} \times \Omega) \cup ([0,T] \times \partial\Omega)$.

Theorem 7.8 (Weak maximum principle)**.** *Let us consider the function $u \in C^{1,2}([0,T] \times \Omega) \cap C(\overline{Q_T})$ for which*

$$\partial_t u - \Delta u \le 0 \qquad \text{on } Q_T.$$

Then, the maximum of the function u is on the boundary Γ_T, meaning that

$$\max_{\overline{Q_T}} u = \max_{\Gamma_T} u.$$

Proof. We prove the statement in two steps: first we show the statement in the case of $\partial_t u - \Delta u < 0$, and in step two we consider the general case.

1. **Step 1:** Let us assume that $\partial_t u - \Delta u < 0$. Our goal is to show that in this case

$$\max_{\overline{Q_T}} u = \max_{\Gamma_T} u.$$

We prove the statement by contradiction: let us assume that there is a point $(t_0, x_0) \in Q_T$ such that $u(t_0, x_0) = \max_{\overline{Q_T}} u$ (so the maximum is not on the boundary). Now we have two consider two cases:

- If $t_0 < T$, then (t_0, x_0) is a local maximum, meaning that $\partial_t u(t_0, x_0) = 0$. Also, $\partial_j^2 u(t_0, x_0) \leq 0$, and from this we have $\Delta u(t_0, x_0) \leq 0$. By substitution, we have

$$\partial_t u(t_0, x_0) - \Delta u(t_0, x_0) \geq 0,$$

which is a contradiction.

- If $t_0 = T$, then $\partial_t u(t_0, x_0) \geq$. Also, $\partial_j^2 u(t_0, x_0) \leq 0$, and from this we have $\Delta u(t_0, x_0) \leq 0$. By substitution, we have

$$\partial_t u(t_0, x_0) - \Delta u(t_0, x_0) \geq 0,$$

which is a contradiction.

2. **Step 2:** Now let us assume that $\partial_t u - \Delta u \leq 0$ on Q_T. We define the function

$$u_\varepsilon(t, x) := u(t, x) - \varepsilon t$$

where $\varepsilon > 0$. Then,

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \partial_t u - \Delta u < 0,$$

so the arguments of step 1 hold for the function u_ε, meaning that

$$\max_{\overline{Q_T}} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon.$$

Since $u_\varepsilon \to u$ uniformly on $\overline{Q_T}$, then

$$\max_{\overline{Q_T}} u = \max_{\Gamma_T} u.$$

Thus, the statement is proved. □

Now we can state a more general result.

Theorem 7.9 (Maximum principle for the heat equation)**.** *Let us consider the bounded function $u \in C^{1,2}((0, T] \times \mathbb{R}^n) \cap C([0, T] \times \mathbb{R}^n)$ for which*

$$\partial_t u - \Delta u \leq 0 \qquad \text{on } (0, T] \times \mathbb{R}^n.$$

Then, the supremum of the function u is at time $t = 0$, meaning that

$$\sup_{[0,T]\times\mathbb{R}^n} u = \sup_{x\in\mathbb{R}^n} u(0, x).$$

Proof. It is enough to prove that

$$\sup_{[0,T]\times\mathbb{R}^n} u \leq \sup_{x\in\mathbb{R}^n} u(0, x). \tag{7.4}$$

Equality (7.4) is a consequence of the following: for all points $(t_0, x_0) \in \overline{\mathbb{R}^{n+1}_+}$ we have

$$u(t_0, x_0) \leq \sup_{x\in\mathbb{R}^n} u(0, x). \tag{7.5}$$

Let us assume that $|u| \leq M$ on $[0, T] \times \mathbb{R}^n$ (such an $M > 0$ number exists since u is bounded) and for a fixed $R > 0$ number let us define

$$v(t, x) = \frac{|x|^2}{2n} + t.$$

Let us also use the notation $Q = (0, T] \times B(x_0, R)$.

Lemma 7.10. *For every point* $(t_0, x_0) \in \overline{\mathbb{R}^{n+1}_+}$ *and for all* $\mu > 0$ *numbers we have*

$$u(t_0, x_0) - \mu v(t_0, x_0) \leq \sup_{x \in \mathbb{R}^n} u(0, x). \tag{7.6}$$

If this lemma holds, then we get inequality (7.5) as $\mu \to 0$, and consequently our statement.

Proof of Lemma 7.10. For equality (7.6) it is enough to prove that

$$\max_{\overline{Q_T}}(u - \mu v) \leq \sup_{x \in \mathbb{R}^n} u(0, x).$$

By simple calculations one can show that $\partial_t v - \Delta v = 0$, meaning that

$$\partial_t(u - \mu v) - \Delta(u - \mu v) \leq 0.$$

Therefore, the weak maximum principle (Theorem 7.8) can be applied to the term $u - \mu v$, meaning that

$$\max_{\overline{Q_T}}(u - \mu v) = \max_{\Gamma_T}(u - \mu v).$$

Then, it is enough to prove that

$$\max_{\Gamma_T}(u - \mu v) \leq \sup_{x \in \mathbb{R}^n} u(0, x).$$

Now we have two cases:

- If we are on $\{0\} \times \overline{B(0, R)}$, then since $v \geq 0$ and $\mu > 0$:

$$\max_{\{0\} \times \overline{B(0,R)}}(u - \mu v) \leq \max_{x \in \overline{B(0,R)}} u(0, x) \leq \sup_{x \in \mathbb{R}^n} u(0, x).$$

- If we are on $[0, T] \times S(0, R)$ (here $S(0, R)$ is the boundary of the disc $B(0, R)$), then since $v(t, x) = \dfrac{R^2}{2n} + t \geq \dfrac{R^2}{2n}$, meaning that

$$\max_{[0,T] \times S(0,R)}(u - \mu v) \leq M - \mu \frac{R^2}{2n}.$$

  If $R \to \infty$, then the right-hand side of the inequality tends to $-\infty$, so there is such a large R value for which

$$\max_{[0,T] \times S(0,R)}(u - \mu v) \leq \sup_{x \in \mathbb{R}^n} u(0, x).$$

Consequently,

$$\max_{\Gamma_T}(u - \mu v) \leq \sup_{x \in \mathbb{R}^n} u(0, x),$$

which proves the lemma. □

Thus, the original statement is proved. □

Other than helping us in the search for the possible places for the maximum of the solution of the heat equation, one can also use the previous result to prove that there is a unique bounded solution of the heat equation.

Theorem 7.11 (Unique bounded solutions of the heat equation)**.** *Let us consider the heat equation*

$$\begin{cases} \partial_t u - \Delta u &= f, \\ u(0, x) &= g, \end{cases}$$

in which $f \in C([0, T] \times \mathbb{R}^n)$ *and* $g \in C(\mathbb{R}^n)$*. Then, it has at most one bounded solution* $u \in C^{1,2}([0, T] \times \mathbb{R}^n)$*.*

Proof. Let us assume that u_1 and u_2 are two bounded solutions of the equation. Then it is easy to see that $u = u_1 - u_2$ is the solution of the homogeneous heat equation, meaning that

$$\begin{cases} \partial_t u - \Delta u &= 0, \\ u(0,x) &= 0. \end{cases}$$

Then, the maximum principle can be applied to the solution, meaning that

$$\sup_{Q^T} u = \sup_{x \in \mathbb{R}^n} u(0,x) = 0.$$

Thus, $u \equiv 0$ (since the solution is non-negative), so $u_1 = u_2$, meaning that the solution is unique. □

In the next chapter we observe elliptic problems, and by using them we start our journey towards Sobolev spaces.

Chapter 8

Boundary-value problems (Elliptic problems)

In the last two chapters we discussed initial-value, or Cauchy-problems. The feature they had in common was that the function we were searching for was depending on time, and also that our space was assumed to be infinite. Although these models form a relatively solid base for our further studies, they are not that useful in practice, since in reality our physical processes usually take place in a bounded domain and not in an infinite space. Because of this, in practice most of the problems which are considered are boundary-value problems, meaning that the processes take place on some bounded (usually connected) domain, and some boundary conditions are given. In this chapter we focus on the easiest class of these, namely elliptic boundary-value problems. This usually means that our solutions do not depend on time: these functions can be thought of as the stationary (i.e. constant in time) solutions of a time-dependent problem.

8.1 Recap, main ideas

In Section 1.2.1, the **stationary heat equation** was defined as

$$-\operatorname{div}(k \operatorname{grad}(u)) = f \qquad \text{inside } \Omega, \tag{8.1}$$

in which $k : \Omega \to \mathbb{R}$ is the thermal conductivity, and $f : \Omega \to \mathbb{R}$ is the source term (these are given functions).

If we apply the choice $k = 1$ to equation (8.1), we get

$$-\Delta u = f, \tag{8.2}$$

which is the **Poisson equation**, and the operator $u \to \Delta u = \operatorname{div}(\operatorname{grad}(u))$ is the Laplace operator.

If $k = 1$ and $f = 0$, we get

$$-\Delta u = 0, \tag{8.3}$$

which is called the **Laplace equation**.

An interesting question is the reason for the minus sign on the left-hand side. The answer is that the operator $-\Delta$ is a positive operator (meaning that $(-\Delta u, u) \geq 0$, in which $(.,.)$ is the inner product defined on the space of functions), and all of its eigenvalues are positive. Since these eigenvalues show up in the series expansion of the solution of these equations (see Section

8.4 for details) and people usually like positive numbers more than negative ones, the equations are defined in this way. (For a physical reason, see Section 1.2.1.)

In the case of boundary-value problems, the boundary conditions (Dirichlet, Neumann, Robin) play an important role: sometimes even the problems are named after them (e.g. Dirichlet problem). From now on, the problems equipped with the first-type, (i.e. Dirichlet), the second-type (i.e. Neumann) and the third-type (i.e. Robin) boundary conditions will be called first-type, second-type and third-type boundary-value problems, respectively.

Our goal in this chapter is to prove some statements concerning the existence and uniqueness of the solutions of equations (8.2) and (8.3). As it was mentioned before, the solutions vary considerably in the cases of the different boundary conditions - however, it is easy to see that the Robin boundary condition is the generalization of the other two, so it is enough to consider just this third one. Also, instead of considering Equation (8.1), we are going to prove our statements for a more general equation, namely

$$-\operatorname{div}(p\,\operatorname{grad}(u)) + qu = f \qquad \text{inside } \Omega, \tag{8.4}$$

in which $p \in C^1(\overline{\Omega})$ and $q \in C(\overline{\Omega})$ are given functions. It turns out that observing this equation is not much harder than equation (8.1), so we will prove theorems concerning this later one (and the desired results are just some special cases of these latter ones).

In the next section we introduce some useful theorems which are necessary in the proofs of our main results.

8.2 Theoretical background

In this section we introduce the main technical tools needed in the main proof of this chapter, namely the formulas of Green. For their proofs, we also need the Gauss-Ostrogradski theorem.

Theorem 8.1 (Gauss-Ostrogradsky theorem)**.** *Let $\Omega \subset \mathbb{R}$ be a compact set with a piece-wise smooth boundary. Suppose that $f : \overline{\Omega} \to \mathbb{R}^n$ is a continuously differentiable function (meaning that its coordinate functions are inside $C^1(\overline{\Omega})$). Then*

$$\int_\Omega \operatorname{div} f = \int_{\partial\Omega} \langle f, \nu \rangle \, d\sigma \tag{8.5}$$

in which ν is the normal of $\partial\Omega$ (and $\langle ., . \rangle$ is the usual inner product).

(The proof is skipped, since I assume that you've learned about this theorem before.)

Remark 8.1. (8.5) can be expressed in the following form: if $g : \overline{\Omega} \to \mathbb{R}$, then

$$\int_\Omega \partial_j g = \int_{\partial\Omega} g\ \nu_j d\sigma \qquad j = 1, \dots n \tag{8.6}$$

in which $\nu = (\nu_1, \dots \nu_n)$ is the normal of $\partial\Omega$ pointing out of Ω. Note that we get the previous result if g is one of the coordinate functions of f and add all of the equations of (8.6).

Now we state the first Green identity.

Theorem 8.2 (First (or asymmetric) Green theorem)**.** *Let $u \in C^2(\overline{\Omega})$, $v \in C^1(\overline{\Omega})$, $p \in C^1(\overline{\Omega})$, and Ω is bounded and has smooth boundary. Then $\forall j = 1, \dots, n$, the following holds:*

$$\int_\Omega v\ \partial_j\left(p\ \partial_j u\right) = -\int_\Omega p\ \partial_j u\ \partial_j v + \int_{\partial\Omega} p\ v\ \partial_j u\ \nu_j\ d\sigma$$

Note that the inner product $\operatorname{grad}(u) \cdot \nu$ is usually denoted by $\partial_\nu u$.

Proof. Let us define $g = v\ (p\ \partial_j u)$. By differentiating it, we get

$$\partial_j g = v\ \partial_j\left(p\ \partial_j u\right) + p\ \partial_j u\ \partial_j v. \tag{8.7}$$

So if we apply the Gauss-Ostrogradski theorem (in the form (8.6)) to g, we get:

$$\int_\Omega \partial_j g = \int_{\partial\Omega} g \cdot \nu_j d\sigma \tag{8.8}$$

Now by substituting the definition of g and (8.7) into (8.8):

$$\int_\Omega v\ \partial_j\left(p\ \partial_j u\right) + \int_\Omega p\ \partial_j u\ \partial_j v = \int_{\partial\Omega} p\ v\ \partial_j u\ \nu_j\ d\sigma,$$

which gives our statement. □

Remark 8.2. The usual form of the first Green theorem (sometimes called Green's first identity or first Green formula) which can be found in textbooks is

$$\int_\Omega v\ \operatorname{div}\left(p\ \operatorname{grad}\ u\right) = -\int_\Omega p\ \operatorname{grad}\ u\ \operatorname{grad}\ v + \int_{\partial\Omega} p\ v\ \partial_\nu u\ d\sigma$$

In some cases $p = 1$, so in these cases we have

$$\int_\Omega v\Delta u = -\int_\Omega \operatorname{grad}\ u\ \operatorname{grad}\ v + \int_{\partial\Omega} v\ \partial_\nu u\ d\sigma$$

Now we state the second Green theorem, which is in fact a consequence of the previous one.

Theorem 8.3 (Second (or symmetric) Green theorem)**.** *Let $u, v \in C^2(\overline{\Omega})$, $p \in C^1(\overline{\Omega})$, and Ω is bounded and has smooth boundary. Then*

$$\int_\Omega v\ \operatorname{div}\left(p\ \operatorname{grad}\ u\right) - u\ \operatorname{div}\left(p\ \operatorname{grad}\ v\right) = \int_{\partial\Omega} p\left(v\partial_\nu u - u\partial_\nu v\right) d\sigma.$$

Proof. By applying the first Green theorem for v and u, and then for u and v:

$$\int_\Omega v\ \operatorname{div}\left(p\ \operatorname{grad}\ u\right) = -\int_\Omega p\ \operatorname{grad}\ u\ \operatorname{grad}\ v + \int_{\partial\Omega} p\ v\ \partial_j u\ \nu_j\ d\sigma$$

$$\int_\Omega u\ \operatorname{div}\left(p\ \operatorname{grad}\ v\right) = -\int_\Omega p\ \operatorname{grad}\ u\ \operatorname{grad}\ v + \int_{\partial\Omega} p\ u\ \partial_j v\ \nu_j\ d\sigma$$

Now by subtracting these from each other, we get the statement. □

Remark 8.3. In the special case of $p = 1$, the second Green identity takes the form

$$\int_\Omega v\ \Delta u - u\Delta v = \int_{\partial\Omega} \left(v\partial_\nu u - u\partial_\nu v\right) d\sigma,$$

which is the usual form found in textbooks.

Remark 8.4. The above two identities also hold in a more general domain, namely if Ω has a Lipschitz boundary (meaning that the boundary can be viewed as a graph of a function with the Lipschitz property), or when $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\operatorname{div}(p \operatorname{grad} u) \in L^1(\Omega)$.

By having our technical background, we can now start to state and prove our main results.

8.3 Classical boundary-value problems

First we define classical boundary-value problems.

Definition 8.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected and open set. Also, $p \in C^1(\overline{\Omega})$, $p > 0$, $q \in C(\overline{\Omega})$, $f \in C(\Omega)$, $g, h, \varphi \in C(\partial\Omega)$ given functions. Then the **classical, third-type boundary-value problem** means the search for such $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ functions, for which

$$\begin{cases} -\operatorname{div}(p \operatorname{grad}(u)) + qu = f & \text{inside } \Omega, \\ gu|_{\partial\Omega} + h\partial_\nu u|_{\partial\Omega} = \varphi. \end{cases} \tag{8.9}$$

If $g = 1$ and $h = 0$, we get the Dirichlet-boundary, and in the case of $g = 0$ and $h = 1$, we get the Neumann one, so it is in fact a generalization of those two.

Remark 8.5. In practice, sometimes it is more useful to define different boundary conditions at different parts of the boundary (e.g. on one side of the domain there is a Dirichlet one, and on some other part we have a Neumann one), but we will not discuss those cases here.

Now we state our main theorem.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected and open set with smooth boundary. Also, $p \in C^1(\overline{\Omega})$, $p > 0$, $q \in C(\overline{\Omega})$, $q \geq 0$, $f \in C(\Omega)$, $g, h, \varphi \in C(\partial\Omega)$ given functions for which $gh \geq 0$ and $g + h \neq 0$. Then:*

- *If $q = g = 0$ does not hold, then there is at most one solution $u \in C^2(\overline{\Omega})$ of the boundary-value problem* (8.9).
- *If $q = g = 0$, then if there is a solution $u \in C^2(\overline{\Omega})$ of the boundary-value problem* (8.9)*, then there are infinitely-many other ones as well, which only differ from this one in a constant (so their difference is a constant function).*

Proof. We prove the theorem in two steps.

1. **Step 1:** If u_1 and u_2 are two solutions of our equation, then by linearity of the equation we get that $u := u_1 - u_2$ is the solution of the boundary-value problem with $f = 0$ and $\varphi = 0$ (since we can subtract them from each other). Because of this, our goal is to show that this u function is zero, meaning that in the case of $f = 0$ and $\varphi = 0$, the problem (8.9) has only the constant zero function as a solution if $u \in C^2(\overline{\Omega})$.

2. **Step 2:** In this part we show that if $u \in C^2(\overline{\Omega})$ is a solution of (8.9) with $f = 0$ and $\varphi = 0$, namely

$$\begin{cases} -\text{div}(p\ \text{grad}(u)) + qu = 0 & \text{inside } \Omega, \\ gu|_{\partial\Omega} + h\partial_\nu u|_{\partial\Omega} = 0, \end{cases}$$

then $u = 0$. For this, let us use the notation

$$Lu := -\text{div}(p\ \text{grad}(u)) + qu.$$

Now we know that $Lu = 0$, meaning that $\int_\Omega u\ Lu = 0$. Also,

$$\int_\Omega u\ Lu = -\int_\Omega u\ \text{div}(p\ \text{grad}(u)) + qu^2 = \tag{8.10}$$

Now let us apply the first Green theorem with $u = v$ to the first term of the right-hand side of (8.10):

$$= \int_\Omega p\ \text{grad}\ u\ \text{grad}\ u - \int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma + \int_\Omega qu^2.$$

Now let us observe the middle term $-\displaystyle\int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma$.

- If $h(x) = 0$, then from the boundary-condition we get that $gu|_{\partial\Omega} = 0$, but since $g + h \neq 0$ and $h = 0$, then $g \neq 0$, meaning that since $gu|_{\partial\Omega} = 0$, then $u|_{\partial\Omega} = 0$. In this case $-\displaystyle\int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma = 0$.
- If $h(x) \neq 0$, then from the boundary-condition:

$$gu|_{\partial\Omega} + h\partial_\nu u|_{\partial\Omega} = 0,$$

$$\partial_\nu u|_{\partial\Omega} = -\frac{g}{h} u|_{\partial\Omega}.$$

In this case the second term is

$$-\int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma = \int_{\partial\Omega} p\ \frac{g}{h}\ u^2\ d\sigma \geq 0,$$

since $p > 0$, and since $gh \geq 0$, we have $\dfrac{g}{h} \geq 0$.

In conclusion,

$$-\int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma \geq 0,$$

which means that

$$0 = \int_\Omega uLu \geq \int_\Omega \left(p\ |\text{grad}\ u|^2 + qu^2\right) \geq 0$$

This can only hold if all of the inequalities are equalities, meaning that grad $u = 0$, so u is a constant function, and we also know that $qu^2 = 0$.

- If $q \neq 0$, then since $qu^2 = 0$, we get that $u = 0$, meaning that $u_1 = u_2$.
- If $q = 0$, then u is some constant function.

Therefore, we have four cases:

- If $q \neq 0$, then $u = 0$, so there is at most one solution.
- If $q = 0$ but $g \neq 0$ and $h = 0$ (special first BVP), then by the previous arguments we have $u|_{\partial\Omega} = 0$ and we know that u is a constant function, so $u = 0$ on Ω (meaning that there is at most one solution).
- If $q = 0$ but $g \neq 0$ and $h \neq 0$ (special third BVP), then we know that u is a constant function. But then $\partial_\nu u = 0$, so since $g \neq 0$ and $h \neq 0$, the boundary condition can only hold if $u|_{\partial\Omega} = 0$, but since u is constant, we have $u = 0$ on Ω (meaning that there is at most one solution).
- If $q = 0$ but $g = 0$ (special second BVP), then we only know that u is constant, so we might have infinitely many solutions, but they only differ from each other in a constant, i.e. $u_1 = u_2 + \text{const}$.

Thus, we got the statement of the theorem. □

The above theorem states that in some cases we can have at most one solution, but it does not talk about the existence of such solutions. In the general case the proof of existence might be hard, but for a given right-hand side it can be done easily (see the corresponding Practice part, namely Practice 9).

Corollary 8.5. *The first and second boundary-value problems can also have only one solution, if $p > 0$ and $q > 0$.*

- *First BVP (Dirichlet): There is at most one solution even for $q = 0$.*
- *Second BVP (Neumann): If $q = 0$, then there can be infinitely-many solutions, but with a constant difference.*

In the next section we define eigenvalue-problems, which are used to solve boundary-value problems.

8.4 Eigenvalue-problems

As mentioned before, eigenvalue-problems (or EVPs) are used to solve boundary-value problems through the method of Fourier, which will be discussed at the end of this section.

Let us define the following linear operator $L : \mathrm{Dom}(L) \subset L^2(\Omega) \to L^2(\Omega)$ with domain

$$\mathrm{Dom}(L) := \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : Lu \in L^2(\Omega), gu|_{\partial\Omega} + h\partial_\nu u|_{\partial\Omega} = 0\}$$

and $Lu := -\mathrm{div}(p\,\mathrm{grad}(u)) + qu$.

Note that in this case the solution of $Lu = f$ means the 3rd boundary-value problem with $\varphi = 0$.

In this section we search for the eigenvalues of this L operator, i.e. search for such numbers $\lambda \in \mathbb{R}$ (eigenvalues) and functions $u \in \mathrm{Dom}(L)$, $u \not\equiv 0$ (eigenfunctions) for which the following holds:

$$Lu = \lambda\, u.$$

Now we state the precise definition:

Definition 8.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected and open set with a smooth boundary. Also, $p \in C^1(\overline{\Omega})$, $p > 0$, $q \in C(\overline{\Omega})$, $g, h \in C(\partial\Omega)$ given functions. Then in the **third-type eigenvalue-problem** (EVP) we search for such $\lambda \in \mathbb{R}$ numbers (eigenvalues) and such $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u \not\equiv 0$ functions (eigenfunctions) for which the following holds:

$$\begin{cases} -\mathrm{div}(p\,\mathrm{grad}(u)) + qu = \lambda u & \text{inside } \Omega, \\ gu|_{\partial\Omega} + h\partial_\nu u|_{\partial\Omega} = 0. \end{cases} \tag{8.11}$$

Here λ is the **eigenvalue**, and u is the **eigenfunction** of operator L (as defined before).

Similarly as before, we can define the first and second EVPs also:

- If $g = 1$ and $h = 0$, then we have the first eigenvalue-problem (and here we only require $u \in C^2(\Omega) \cap C(\overline{\Omega})$, since there is no derivative in the boundary condition).
- If $g = 0$ and $h = 1$, then we have the second eigenvalue-problem.

Now we can state the main theorem of this section:

Theorem 8.6. *Under the above conditions, and also assuming that $q \geq 0$, $gh \geq 0$ and $g + h \neq 0$, the following holds:*

1. *L is a symmetric and positive operator. It can have at most countably-many (possibly infinitely-many) eigenvalues, these are non-negative, and the eigenfunctions corresponding to the different eigenvalues are orthogonal (in the usual inner product defined for functions).*
2. *$\lambda = 0$ can only be an eigenvalue if $q = 0$ and $g = 0$ (the 2nd EVP) and then the eigenfunctions are constant functions.*

Proof. We prove the statements in four steps.

1. **Step 1:** In this step we prove that L is symmetric, i.e. $(Lu, v)_{L^2} = (u, Lv)_{L^2}$ (in which $(.,.)_{L^2}$ is the inner product defined in the L^2-space, namely $(f, g)_{L^2(\Omega)} = \int_\Omega fg$.)

   $$(Lu, v)_{L^2} - (u, Lv)_{L^2} =$$

   Now we use the second Green theorem:

   $$= \int_{\partial\Omega} p\,(v\partial_\nu u - u\partial_\nu v)\,d\sigma$$

   Then, by similar arguments as in the previous proof:

   $$-pv\partial_\nu u = \begin{cases} 0 & \text{if } h(x) = 0, \\ p\dfrac{g}{h}uv & \text{if } h(x) \neq 0 \end{cases}$$

   $$-pu\partial_\nu v = \begin{cases} 0 & \text{if } h(x) = 0, \\ p\dfrac{g}{h}uv & \text{if } h(x) \neq 0 \end{cases}$$

   Then this means that

   $$(Lu, v)_{L^2} - (u, Lv)_{L^2} = \int_{\partial\Omega} p\,(v\partial_\nu u - u\partial_\nu v)\,d\sigma = 0,$$

   so $(Lu, v)_{L^2} = (u, Lv)_{L^2}$, meaning that L is symmetric.

2. **Step 2:** In this step we prove that L is positive, meaning that $(Lu, u)_{L^2} \geq 0$. However, in the previous proof we showed that

$$0 \leq \int_\Omega u\, Lu = (Lu, u)_{L^2},$$

so L is positive.

3. **Step 3:** In this step we show the properties of the eigenvalues and the eigenfunctions. For this, we use a theorem from functional analysis, which states that positive operators have indeed non-negative eigenvalues, and the eigenfunctions are orthogonal to each other. From this orthogonality property we also get that there can be at most countably many eigenfunctions[1], meaning that there are countably many eigenvalues also.

4. **Step 4:** Here we prove part 2 of the theorem. If $\lambda = 0$ is an eigenvalue, then it means that $Lu = 0 \cdot u = 0$, so $(Lu, u)_{L^2} = 0$. Then by the previous proof we know that in this case $q = 0$ and $g = 0$ (since in all the other cases we have $u \equiv 0$ which is not an eigenfunction), meaning that u is constant.

This gives our statement. □

Application of the eigenvalue-problems: the method of Fourier

In this section we show the reason the EVPs are important in the solution of BVPs. The method presented here, usually referred to as the **method of Fourier** is a pretty useful tool to find solutions of BVPs for some special f functions.

Suppose that $Lu = f$ inside Ω, and we have a homogeneous boundary condition, L has countably-many non-zero λ_k eigenvalues, and the eigenfunctions form a complete orthogonal system in the function-space. (Note that all of these hold if the assumptions of the previous theorem are fulfilled.)

Then we search for the solution of the BVP in the form $u = \sum_{k=1}^{\infty} \xi_k u_k$, in which $u_k,\ k = 1, 2, \ldots$ are the eigenfunctions of L and the values ξ_k are some unknown constants. If the above series can be differentiated term-by-term, then the following holds (since L is linear):

$$Lu = \sum_{k=1}^{\infty} \xi_k L u_k = \sum_{k=1}^{\infty} \xi_k \lambda_k u_k.$$

Let us also write up f as a (probably infinite) sum of the eigenfunctions:

$$f = \sum_{k=1}^{\infty} c_k u_k.$$

(Such a series exists if $u_k,\ k = 1, 2, \ldots$ forms a complete orthogonal base in the function space, which is the case of the previous theorem).

[1] This comes from the fact that the space $L^2(\Omega)$ is separable, meaning that there can be at most countably many linearly independent elements in it. The proof of this property can be found in any good functional analysis textbook.

Then by our equation $Lu = f$, we have:

$$\sum_{k=1}^{\infty} \xi_k \lambda_k u_k = \sum_{k=1}^{\infty} c_k u_k.$$

This can only hold if all of the coefficients are the same, meaning that $\xi_k \lambda_k = c_k$ for all indexes of k and consequently $\xi_k = \frac{c_k}{\lambda_k}$, so our solution is in the form

$$u = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} u_k.$$

Note that all of the terms on the right-hand side can be calculated, meaning that we got a formula for our solution. (Of course, in practice the search for λ_k, u_k or even for c_k might be difficult.) Also, since the solution is an infinite sum, in applications we will not get the precise value of the solution (we would have to add up infinitely many terms), but usually the summation is stopped after a given number of terms (since usually the terms with larger indices are smaller, which can be achieved if they are listed in a descending order).

In the next chapter we define Green's functions, and using them we find another method to construct our solutions.

Chapter 9

Green's functions (skipped in 2024)

We are going to skip this whole chapter.

As it was mentioned at the end of the previous chapter, a quite useful method of constructing solutions of PDEs involves Green's functions.

9.1 Green's third theorem

Before we can state our main result, we will need another result of Green.

Theorem 9.1 (Green's third theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary, and $x_0 \in \Omega$. Let $y \to w(x_0, y) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be an arbitrary function, for which $\Delta w \in L^1(\Omega)$, and consider the following function*

$$y \to F(x_0, y) = E(x_0 - y) - w(x_0, y), \qquad (y \in \overline{\Omega})$$

in which E is the fundamental solution of the problem, namely

$$E(x) := \begin{cases} -\dfrac{1}{(n-2)\omega_n} \dfrac{1}{|x|^{n-2}} & , \qquad \text{if } n \geq 3, x \in \mathbb{R}^n \setminus \{0\}, \\ -\dfrac{1}{2\pi} \log\left(\dfrac{1}{|x|}\right) & , \qquad \text{if } n = 2, x \in \mathbb{R}^n \setminus \{0\}, \end{cases}$$

as it was defined in Chapter 5. Then for any $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, for which $\Delta u \in L^1(\Omega)$, the following holds:

$$\begin{aligned} u(x_0) = &\int_{\partial\Omega} (F(x_0, y)\partial_\nu u(y) - u(y)\partial_\nu F(x_0, y))\, d\sigma_y - \\ &- \int_\Omega (F(x_0, y)\Delta u(y) - u(y)\Delta F(x_0, y))\, dy. \end{aligned} \tag{9.1}$$

Proof. We prove the statement in three steps.

1. **Step 1:** Let ε such a small value for which $\overline{B(x_0, \varepsilon)} \subset \Omega$. Let us apply the second Green theorem to the function $y \to F(x_0, y)$ on the domain $\Omega \setminus \overline{B(x_0, \varepsilon)}$ (see Figure 9.1[1])):

$$\begin{aligned} &\int_{\Omega \setminus \overline{B(x_0,\varepsilon)}} (F(x_0, y)\Delta u(y) - u(y)\Delta F(x_0, y))\, dy = \\ &= \int_{\partial\Omega} (F(x_0, y)\partial_\nu u(y) - u(y)\partial_\nu F(x_0, y))\, d\sigma_y + \\ &\quad + \int_{S(x_0,\varepsilon)} (F(x_0, y)\partial_\nu u(y) - u(y)\partial_\nu F(x_0, y))\, d\sigma_y. \end{aligned} \tag{9.2}$$

[1]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

Let us assume that

$$\lim_{\varepsilon\to 0+}\int_{\Omega\setminus\overline{B(x_0,\varepsilon)}}(F(x_0,y)\Delta u(y)-u(y)\Delta F(x_0,y))\,dy= \\ =\int_{\Omega}(F(x_0,y)\Delta u(y)-u(y)\Delta F(x_0,y))\,dy, \tag{9.3}$$

and

$$\lim_{\varepsilon\to 0+}\int_{S(x_0,\varepsilon)}(F(x_0,y)\partial_\nu u(y)-u(y)\partial_\nu F(x_0,y))\,d\sigma_y=u(x_0) \tag{9.4}$$

holds. Then if we take the limit of (9.2) as $\varepsilon\to 0+$, then we get (9.1) which we wanted to prove.

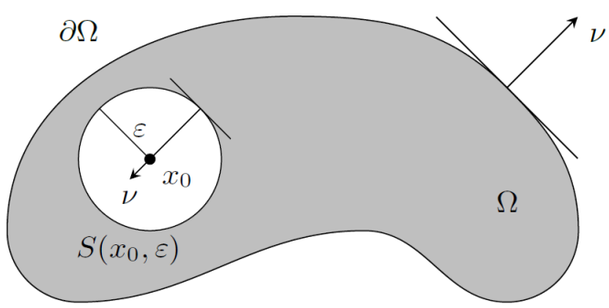


Figure 9.1: The domain with the "hole" in it.

In steps 2 and 3 we prove the assumptions (9.3) and (9.4).

2. **Step 2:** (9.3) is true if the function

$$y\to F(x_0,y)\Delta u(y)-u(y)\Delta F(x_0,y)$$

is integrable on Ω. Since F is continuous on $\overline{\Omega}$, it is bounded, and Δu is also integrable on Ω, so the first term is integrable. For the second one, we know that u is continuous on $\overline{\Omega}$, and since E is a fundamental solution, then in the case $y\neq x_0$ we have $\Delta(y\to E(x_0-y))=0$, which means that $\Delta F(x_0,y)=-\Delta w(x_0,y)$, which is integrable, so the second term is also integrable, therefore the limit makes sense.

3. **Step 3:** Now we prove (9.4). It is easy to see that

$$\int_{S(x_0,\varepsilon)}(F(x_0,y)\partial_\nu u(y)-u(y)\partial_\nu F(x_0,y))\,d\sigma_y= \\ =-\int_{S(x_0,\varepsilon)}E(x_0-y)\partial_\nu u(y)d\sigma_y-\int_{S(x_0,\varepsilon)}w(x_0,y)\partial_\nu u(y)d\sigma_y+ \\ +\int_{S(x_0,\varepsilon)}u(y)\partial^y_\nu E(x_0-y)d\sigma_y+\int_{S(x_0,\varepsilon)}u(y)\partial^y_\nu w(x_0,y)d\sigma_y. \tag{9.5}$$

For the first term of (9.5), we know that since u is bounded and by the definition of E, we have

$$\left|\int_{S(x_0,\varepsilon)}E(x_0-y)\partial_\nu u(y)d\sigma_y\right|\le\int_{S(x_0,\varepsilon)}\frac{C_1}{\varepsilon^{n-1}}d\sigma_y=\frac{C_1}{\varepsilon^{n-2}}\omega_n\varepsilon^{n-1}\longrightarrow 0\quad\text{as }\varepsilon\to 0+$$

if $n\ge 3$, and in the case of $n=2$:

$$\left|\int_{S(x_0,\varepsilon)}E(x_0-y)\partial_\nu u(y)d\sigma_y\right|\le C_1\varepsilon|\log\varepsilon|\longrightarrow 0\quad\text{as }\varepsilon\to 0+.$$

Similarly, the second and fourth terms in (9.5) also tend to zero when $\varepsilon\to 0+$, since the area of the surface on which we integrate shrinks.

For the third term of (9.5), let us observe that the normal of our domain on$S(0,\varepsilon)$ points to the center of this ball (see Figure 9.1), so

$$\partial_\nu(y \to E(x_0-y))|_{S(x_0,\varepsilon)} = \partial_{|x_0-y|}(y \to E(x_0-y))|_{S(x_0,\varepsilon)} = \frac{1}{\omega_n \varepsilon^{n-1}},$$

so consequently

$$\int_{S(x_0,\varepsilon)} u(y)\partial_\nu^y E(x_0-y)d\sigma_y = \frac{1}{\omega_n\varepsilon^{n-1}}\int_{S(x_0,\varepsilon)} u(y)d\sigma_y,$$

meaning that if $\varepsilon \to 0+$, then it tends to $u(x_0)$, so the assumption is proved.

Thus, the statement is proved. □

Remark 9.1. If w is such a function for which $\Delta F = 0$ outside of x_0 and $F = 0$ on $\partial\Omega$, then

$$u(x_0) = -\int_{\partial\Omega} u(y)\partial_\nu F(x_0,y)d\sigma_y - \int_\Omega F(x_0,y)\Delta u(y)dy$$

This would mean that we have a formula for u in this case (since $u|_{\partial\Omega}$ and Δu are given). Now our goal is to find such F functions (or, such w for which these hold).

9.2 Green's functions and their properties

In this section we define Green's functions, and also prove some of their properties.

9.2.1 The case of the First Boundary-value problem

Consider the first boundary-value problem defined on $\Omega \subset \mathbb{R}^n$ which is a bounded domain with smooth boundary:

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases} \tag{9.6}$$

in which $f \in C(\Omega)$, $\varphi \in C(\partial\Omega)$ given functions.

Definition 9.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that for any fixed $x \in \Omega$, there is a function $v_x \in C^2(\Omega) \cup C^1(\overline{\Omega})$ for which

$$\begin{cases} -\Delta v_x = 0 & \text{on } \Omega, \\ v_x(y) = E(x-y) & y \in \partial\Omega, \end{cases}$$

in which E is the fundamental solution of problem (9.6).

Let us use the notation $w(x,y) = v_x(y)$ $(x \in \Omega, y \in \overline{\Omega})$. Then the function

$$G(x,y) = E(x-y) - w(x,y) \qquad (x \in \Omega, y \in \overline{\Omega})$$

is called the **Green's function** of problem (9.6).

Now we can state the main result of this section.

Theorem 9.2 (Green's representation theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that (9.6) has a Green's function. Then if $u \in C^2(\Omega) \cup C^1(\overline{\Omega})$ is a solution of (9.6) with $f \in C(\overline{\Omega})$, then*

$$u(x) = -\int_{\partial\Omega} \varphi(y)\partial_\nu^y G(x,y)d\sigma_y + \int_\Omega G(x,y)f(y)dy.$$

Proof. Let us apply Green's 3rd theorem with $F = G$. Consequently,

$$\Delta F = \Delta G = \Delta(E - w) = 0$$

and

$$F|_{\partial\Omega} = (E - w)\,|_{\partial\Omega} = (E - E)|_{\partial\Omega} = 0.$$

In the first equation we used the fact that $\Delta E = 0$, which can be proved the following way (not part of the exam material).

If $n = 2$, then $E(x,y) = -\dfrac{1}{2\pi}\log\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$, so

$$2\pi\partial_x E(x,y) = \partial_x\left(\log\sqrt{x^2+y^2}\right) = \frac{1}{2}\partial_x\left(\log\left(x^2+y^2\right)\right) = \frac{x}{x^2+y^2},$$

and therefore

$$2\pi\partial_x^2 E(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and by symmetry,

$$2\pi\partial_y^2 E(x,y) = \frac{x^2-y^2}{(y^2+x^2)^2},$$

which means that $\Delta E = \partial_x^2 E(x,y) + \partial_y^2 E(x,y) = 0$.

If $n \geq 3$, then $E(x,y) = -\dfrac{1}{(n-2)\omega_n}\dfrac{1}{|x|^{n-2}}$, meaning that

$$\omega_n \partial_{x_i} E = x_i|x|^{-n}$$

and

$$\omega_n \partial_{x_i}^2 E = |x|^{-n-2}\left(|x|^2 - nx_i^2\right),$$

consequently

$$\omega_n \Delta E = \omega_n \sum_{j=1}^n \partial_{x_i}^2 E = \omega_n |x|^{-n-2}\left(n|x|^2 - n|x|^2\right) = 0,$$

which gives the statement. □

So if we know the Green's function of a problem, then we have now a formula for our solution.

Now we state some properties of the Green's functions.

Theorem 9.3 (Properties of Green's functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Then the following properties hold:*

(i) $\Delta_y G(x,y) = 0 \qquad$ *if* $x \in \Omega, y \in \Omega, y \neq x$,

(ii) $G(x,y) = 0 \qquad$ *if* $x \in \Omega, y \in \partial\Omega$,

(iii) $G(x,y) = G(y,x) \qquad$ *if* $x \in \Omega, y \in \Omega, y \neq x$.

Proof. (i) By definition:

$$\Delta_y G(x,y) = \Delta_y\left[E(x-y) - w(x,y)\right] = 0.$$

(ii) By definition:

$$G(x,y)|_{y\in\partial\Omega} = (E(x-y) - w(x,y))\,|_{y\in\partial\Omega} = (E(x-y) - E(x-y))\,|_{y\in\partial\Omega} = 0.$$

(iii) (Not part of the exam material.)

Let $x_1, x_2 \in \Omega$, $x_1 \neq x_2$ to be fixed points and choose ε in a way that the balls $\overline{B(x_1,\varepsilon)} \subset \Omega$ and $\overline{B(x_2,\varepsilon)} \subset \Omega$ are disjoint. Let us then apply Green's 2nd formula for the domain $\Omega \setminus \left(\overline{B(x_1,\varepsilon)} \cup \overline{B(x_2,\varepsilon)}\right)$ (see Figure 9.2[2]):

$$\begin{aligned}&\int_{\Omega\setminus\left(\overline{B(x_1,\varepsilon)}\cup\overline{B(x_2,\varepsilon)}\right)} (G(x_2,y)\Delta_y G(x_1,y) - G(x_1,y)\Delta_y G(x_2,y))\,dy = \\ &= \int_{\partial\Omega\cup S(x_1,\varepsilon)\cup S(x_2,\varepsilon)} (G(x_2,y)\partial_\nu G(x_1,y) - G(x_1,y)\partial_\nu G(x_2,y))\,d\sigma_y.\end{aligned} \tag{9.7}$$

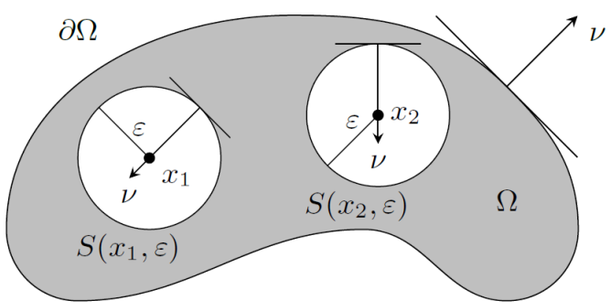


Figure 9.2: The domain with the two "holes" in it.

By property (i), we know that the left-hand side of (9.7) is zero, and by property (ii) on the right-hand side we have

$$\int_{\partial\Omega} (G(x_2,y)\partial_\nu G(x_1,y) - G(x_1,y)\partial_\nu G(x_2,y))\,d\sigma_y = 0,$$

so it means that

$$\int_{S(x_1,\varepsilon)\cup S(x_2,\varepsilon)} (G(x_2,y)\partial_\nu G(x_1,y) - G(x_1,y)\partial_\nu G(x_2,y))\,d\sigma_y = 0.$$

By the proof of the third Green theorem, if we take the limit of this term as $\varepsilon \to 0+$, then we have

$$G(x_2,x_1) - G(x_1,x_2) = 0,$$

which is the statement we wanted to prove.

This concludes the proof. □

9.2.2 The case of the Third Boundary-Value Problem

In this section we consider a special case of the third boundary-value problem, namely

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \partial_\nu u|_{\partial\Omega} + h\ u|_{\partial\Omega} = \varphi, \end{cases} \tag{9.8}$$

in which $f \in C(\Omega)$, $\varphi, h \in C(\partial\Omega)$ given functions, $h > 0$.

[2]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

Definition 9.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that for any fixed $x \in \Omega$, there is a function $v \in C^2(\Omega) \cup C^1(\overline{\Omega})$, for which

$$\begin{cases} -\Delta v = 0 & \text{on } \Omega, \\ \partial_\nu v(y) + h(y)\, v(y) = \partial_\nu^y E(x-y) + h(y)E(x-y) & y \in \partial\Omega, \end{cases}$$

in which E is the fundamental solution of (9.8).

Let us use the notation $w(x,y) = v(y)$ $(x \in \Omega, y \in \overline{\Omega})$. Then the function

$$G(x,y) = E(x-y) - w(x,y) \qquad (x \in \Omega, y \in \overline{\Omega})$$

is called the **Green's function** of problem (9.8).

Now we can state the main result of this section.

Theorem 9.4 (Green's representation theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that* (9.8) *has a Green's function. Then if $u \in C^2(\Omega) \cup C^1(\overline{\Omega})$ is a solution of* (9.8) *with $f \in C(\overline{\Omega})$, then*

$$u(x) = -\int_{\partial\Omega} \varphi(y)\partial_\nu^y G(x,y)d\sigma_y + \int_\Omega G(x,y)f(y)dy.$$

The proof is skipped (it is similar to the previous case).

So if we know the Green's function of a problem, then we have now a formula for our solution.

Remark 9.2. The theorem about the properties of the Green's functions also hold, but in (ii) we have

$$\partial_\nu^y G(x,y) + h(y)G(x,y) = 0 \qquad (y \in \partial\Omega).$$

9.3 Construction of Green's function

In the previous two sections we saw that by using Green's functions we can construct our solution. However, the search for a Green's function might be hard. It can be shown that if our domain has a smooth boundary, then the problem corresponding to it has a Green's function. Here we show two examples in which this construction is not that complicated. The main tool in both of them is the use of a reflection defined on the boundary in a smart way.

9.3.1 Poisson's formula on a sphere

Let $S(0,R)$ be the surface of the ball centered at the origin with radius R, and $B(0,R)$ be the open ball itself. Then the **inversion** on this ball is the map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ defined as

$$x \to \tilde{x} := \frac{R^2}{|x|^2}x.$$

So this means that $\tilde{x}$ is such a point on the line $\overrightarrow{0x}$ for which $|x| \cdot |\tilde{x}| = R^2$ (see Figure 9.3[3]). Then all of the points inside the ball are mapped outside of it, and the points on the surface are not changed.

[3]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.

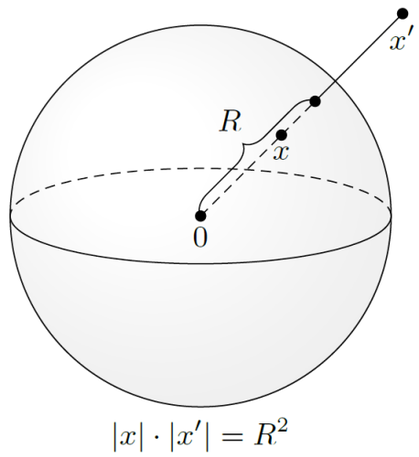


Figure 9.3: The map of inversion (here x' is the map of x).

Now we search for the Green's function of the problem defined on this ball (so $\Omega = B(0,R)$). For this, we will need a $v(y)$ function for which the following holds:

$$\begin{cases} -\Delta v = 0 & \text{on } B(0,R), \\ v = E(x-y) & y \in S(0,R). \end{cases}$$

Our main idea is that $v(y) = E(x-y)$ is almost a good choice, since $\Delta_y E(x-y) = 0$ outside of the point $y = x$. The problem is at point x, since the fundamental solution $E(x)$ was not defined for $x = 0$. Because of this, we use an inversion: we will transform our points outside of our sphere!

Let $v(y) := E(\gamma(\tilde{x} - y))$, in which γ is a suitable constant depending on x which will be defined later, and $\tilde{x}$ is the map of x using the inversion defined before. For this function $\Delta v = 0$ on $B(0,R)$, since E is a fundamental solution, and if $x \in B(0,R)$, then $\tilde{x} \neq 0$ (since if $\tilde{x} = 0$, then we have $0 \cdot |x| = R^2$ but this cannot hold.)

Our goal now is to choose γ in a way that $v(y) = E(x-y)$ on $S(0,R)$.

Proposition 9.5. *If $\tilde{x}$ is the map of x using the inversion defined before and $y \in S(0,R)$, then*

$$|x-y| = \frac{|x|}{R}|\tilde{x}-y|.$$

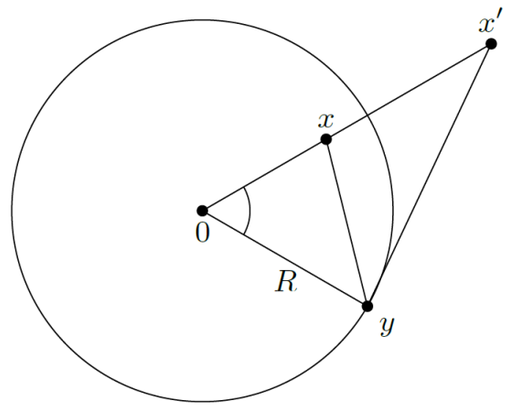
Proof. We prove the statement using some geometric considerations. Let us observe Figure 9.4[4]! Here the $0xy$ and the $0y\tilde{x}$ triangles are similar, since the angles $x0y$ and $\tilde{x}0y$ are the same, and the ratio of sides $0x$, $0y$ and sides $0y$ and $0\tilde{x}$ are the same, namely

$$\frac{|x|}{R} = \frac{R}{|\tilde{x}|}$$

which is a consequence of $|\tilde{x}| \cdot |x| = R^2$. Also, the side $0y$ is part of both of them, so they are indeed similar. This also means that the sides at the opposite of angle $x0y$ have also the same ratio, so

$$\frac{|x|}{R} = \frac{|x-y|}{|\tilde{x}-y|},$$

[4]Source: Besenyei-Komornik-Simon: Parciális Differenciálegyenletek. Typotex, 2013.



$$|x| : R = R : |x'| = |x - y| : |x' - y|$$

Figure 9.4: The triangles used in the proof.

from which we get

$$|x - y| = \frac{|x|}{R}|\tilde{x} - y|,$$

which was the statement we wanted to prove. □

By this proposition we get that if $\gamma = \dfrac{|x|}{R}$, then

$$v(y) = E(\gamma(\tilde{x} - y)) = E\left(\frac{|x|}{R}(\tilde{x} - y)\right).$$

In conclusion, the choice $v(y) = E\left(\dfrac{|x|}{R}(\tilde{x} - y)\right)$ is a good one, since $v \in C^2(B(0,R)) \cup C^1(\overline{B(0,R)})$, $\Delta v = 0$ on $B(0,R)$ and $v(y) = E(x - y)$ on $S(0,R)$ (by the previous proposition).

Note that if $x = 0$, then the above arguments do not work. However, if we choose $v(y) = E(R)$, i.e.

$$v(y) = \begin{cases} -\dfrac{1}{2\pi}\log|R| & \text{if } n = 2, \\ \dfrac{1}{(n-2)\omega_n|R|^{n-2}} & \text{if } n \geq 3, \end{cases}$$

in which ω_n is the surface of the n-dimensional sphere, then $\Delta v = 0$, and also if $y \in S(0,R)$, then $v(y) = E(R) = E(0 - y) = E(|y|) = E(R)$.

In conclusion, we can state the following proposition.

Proposition 9.6. *The Green function for the $B(0,R)$ sphere has the form*

$$G(x,y) = E(x - y) - E\left(\frac{|x|}{R}(\tilde{x} - y)\right) \qquad \text{if } x \neq 0, y \in \overline{B(0,R)},$$

and

$$G(0,y) = E(-y) - E(R) \qquad y \in \overline{B(0,R)},$$

in which E is the fundamental solution of the Laplace equation.

Note that in Green's representation theorem we also need $\partial_\nu^y G(x,y)$.

Proposition 9.7. *If G is the Green function of $B(0,R)$, then*

$$\partial_\nu^y G(x,y) = \frac{-1}{\omega_n}\frac{R^2-|x|^2}{|x-y|^n}$$

for $x \in B(0,R)$ and $y \in S(0,R)$.

The proof is just simple calculations.

Combining the previous two results, we get our main result.

Corollary 9.8. *If $f = 0$, then the solution of the Poisson equation has the form*

$$u(x) = \frac{R^2-|x|^2}{R\omega_n}\int_{S(0,R)}\frac{\varphi(y)}{|x-y|^n}d\sigma_y \qquad (x \in B(0,R)). \tag{9.9}$$

*This is called the **Poisson formula for the sphere**. Also, sometimes*

$$K(x,y) = \frac{R^2-|x|^2}{R\omega_n}\frac{1}{|x-y|^n}$$

*is called the **Poisson kernel**.*

Theorem 9.9. *Let $\varphi \in C(S(0,R))$ and u be as defined in (9.9). Then $u \in C^2(B(0,R)) \cup C^1(\overline{B(0,R)})$ and it is the solution of*

$$\begin{cases} -\Delta u = 0, & \text{in } B(0,R), \\ u|_{S(0,R)} = \varphi. \end{cases}$$

(Proof is skipped.)

9.3.2 Poisson's formula on a half-space

In this section we construct the Green's function for a half space, meaning that our domain will be the set

$$\mathbb{R}^n_+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}$$

So we search for the solution (and also the Green's function) for the problem

$$\begin{cases} -\Delta u = 0, & \text{in } \mathbb{R}^n_+, \\ u|_{\partial\mathbb{R}^n_+} = \varphi. \end{cases}$$

As in the previous section, we are also going to define a reflection, now to the hyperplane $\partial\mathbb{R}^n_+ := \{(x_1,\dots,x_{n-1},0)\}$, namely for $x = (x_1,\dots x_n)$ we have

$$x \to \tilde{x} = (x_1,\dots,x_{n-1},-x_n).$$

Then it is easy to see that for all $y \in \mathbb{R}^n_+$, $|x-y| = |\tilde{x}-y|$, meaning that

$$G(x,y) = E(x-y) - E(\tilde{x}-y)$$

is a good choice for a Green's function, since for all $x \in \mathbb{R}^n_+$, $\Delta_y G(x,y) = 0$ on $\mathbb{R}^n_+$, and also $G(x,y) = 0$ on $\partial\mathbb{R}^n_+$ (since the reflection of a point on the boundary is itself).

We also know that on the boundary, the normal is in the form $\nu = (0, \dots, 0, -1)$, so if $n \geq 3$, then

$$\partial_\nu^y G(x,y) = -\partial_{y_n} G(x,y) = \frac{1}{\omega_n}\left(\frac{y_n - x_n}{|x-y|^n} - \frac{y_n - \tilde{x}_n}{|\tilde{x}-y|^n}\right) = \frac{-2}{\omega_n}\frac{x_n}{|x-y|^n}.$$

Here we used the fact that if $n \geq 3$, then

$$\partial_{y_i} E(x-y) = \partial_{y_i}\left(-\frac{1}{(n-2)\omega_n}\frac{1}{|x-y|^{n-2}}\right) =$$

$$= -\frac{1}{(n-2)\omega_n}\partial_{y_i}\left(\frac{1}{(\sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2})^{n-2}}\right) =$$

$$= \frac{1}{(n-2)\omega_n}\frac{(n-2)}{|x-y|^{n-1}}\frac{1}{2}\frac{1}{|x-y|}2(x_i-y_i) = \frac{1}{\omega_n}\frac{x_i-y_i}{|x-y|^n}.$$

If $n = 2$, then we can say that $\omega_2 = 2\pi$ and then we get a similar formula:

$$\partial_{y_i} E(x-y) = \partial_{y_i}\left(-\frac{1}{2\pi}\log\left(\frac{1}{|x-y|}\right)\right) =$$

$$= -\frac{1}{2\pi}\partial_{y_i}\left(\log\left(\frac{1}{\sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}}\right)\right) =$$

$$= \frac{1}{2\pi}|x-y|\frac{-1}{|x-y|^2}\frac{1}{2}\frac{1}{|x-y|}2(x_i-y_i) = \frac{1}{2\pi}\frac{x_i-y_i}{|x-y|^2}.$$

From this we get that our solution is in the form

$$u(x) = \frac{2x_n}{\omega_n}\int_{\partial\mathbb{R}^n_+}\frac{\varphi(y)}{|x-y|^n}d\sigma_y. \tag{9.10}$$

This is called the **Poisson's formula for the half-space**. Also,

$$K(x,y) = \frac{2x_n}{\omega_n}\frac{1}{|x-y|^n}$$

is called the **Poisson kernel**.

Theorem 9.10. *Let $\varphi \in C(\mathbb{R}^{n-1}) \cup L^\infty(\mathbb{R}^{n-1})$. Then for the u defined as (9.10), we have $u \in C^2(\mathbb{R}^n_+) \cup C(\overline{\mathbb{R}^n_+}) \cup L^\infty(\mathbb{R}^n_+)$, and u is the solution of the problem*

$$\begin{cases} -\Delta u = 0, & \text{in } \mathbb{R}^n_+, \\ u|_{\partial\mathbb{R}^n_+} = \varphi. \end{cases}$$

In the next chapter we define Sobolev spaces, and then observe the general solutions of the elliptic boundary problems.

Chapter 10

Sobolev spaces

In this chapter we define Sobolev spaces, and show some of their properties. They are not only useful in the theory of Partial Differential Equations, but also in Stochastic Differential Equations. However, here we only focus on their applications for PDEs.

As it was mentioned in previous chapters, usually the weak form of a PDE is more useful than the classical form: the first one means that we take the derivatives in the equation not in the classical sense, but rather in the distributional sense. The derivative of a function f in the distributional sense meant that we consider the regular distribution T_f corresponding to f, and then we differentiate this distribution. If we get a regular distribution after the derivation, namely $\partial T_f = T_g$, then we can say that function g is the distributional derivative of function f.

Now a natural question arises: which functions have a distributional derivative such that it is a regular distribution? We defined regular distributions for all functions $f \in L^1_{loc}$, but we have also seen that the derivative of regular distributions is usually not a regular distribution (e.g. $\partial T_H = \delta_0$ in which H is the Heaviside function), so they do not have a distributional derivative as defined above, at least not such a derivative which is a function. Sobolev spaces solve this problem: these will be the spaces in which all the functions have distributional derivatives, and these derivatives are also functions.

10.1 Definition of $H^k(\Omega)$ and $H^k_0(\Omega)$

As it was mentioned in the introduction, our goal is to define a space in which all the functions have nice distributional derivatives. For this, we will define some function spaces, and then show that they are indeed those which we were looking for.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected domain which is bounded.

Definition 10.1. Consider the space $C^k(\overline{\Omega})$ ($k \in \mathbb{N}$). Let us define the following inner product on this space (it can be shown that it is indeed an inner product):

$$\langle f, g \rangle := \sum_{|\alpha| \leq k} \int_\Omega (\partial^\alpha f)\,(\partial^\alpha g) \tag{10.1}$$

Then the completion of the space $C^k(\overline{\Omega})$ with this inner product is called a **Sobolev space**, and is denoted by $H^k(\Omega)$. This is then a Hilbert space.

Remark 10.1. A quick functional analysis recap, so the above definition makes more sense:

- Every inner product induces a norm by the rule $\|x\| := \sqrt{\langle x, x \rangle}$. The norm induced by the inner product (10.1) is

$$\|f\|_{H^k(\Omega)} = \left[\int_\Omega \left(|f|^2 + \sum_{|\alpha| \leq k} |\partial^\alpha f|^2 \right) \right]^{1/2} \tag{10.2}$$

- A *Cauchy-sequence* is a sequence x_n for which for every $\varepsilon > 0$ there exists a bound N after which for all indexes $n, m > N$ we have $\|x_n - x_m\| < \varepsilon$. This basically means that the elements get closer and closer to each other after some time.

- Every convergent sequence is a Cauchy-sequence (if it is convergent, then the elements should get closer after a while), but not every Cauchy-sequence is convergent (because if might happen that the element the sequence is getting closer and closer to is not part of our space). The normed spaces in which all Cauchy-sequences are convergent are called *complete*, and they are also called *Banach spaces*. If there is also an inner product defined on this space, then a complete inner product space is called a *Hibert space*.

- A *completion* of an inner product space X means that we take a larger space Y, in which all of the Cauchy sequences defined on X are convergent (so basically $Y = X \cup Z$, in which Z is the set of the "limits" of the Cauchy sequences defined on X).

Remark 10.2. Note that the inner product defined as (10.1) can be also written as

$$\langle f, g \rangle := \sum_{|\alpha| \leq k} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}, \tag{10.3}$$

in which $(.,.)_{L^2(\Omega)}$ is the usual inner product defined on space $L^2(\Omega)$, namely

$$(f, g)_{L^2(\Omega)} = \int_\Omega fg.$$

The norm (10.2) can be also written in a shorter form using the usual $L^2(\Omega)$-norm, namely

$$\|f\|_{H^k(\Omega)} = \left[\|f\|^2_{L^2(\Omega)} + \sum_{|\alpha| \leq k} \|\partial^\alpha f\|^2_{L^2(\Omega)} \right]^{1/2} \tag{10.4}$$

A natural question: what is the connection between the Sobolev spaces defined above, and our goal, namely the space in which all of the functions have "nice" distributional derivatives? The next theorem answers this question.

Theorem 10.1. *The following two are equivalent.*

(a) $f \in H^k(\Omega)$

(b) The distributional derivatives $\partial^\alpha f$, $|\alpha| \le k$ *exist for* $f \in L^2(\Omega)$, *these derivatives are inside* $L^2(\Omega)$, *and there is a sequence* $f_j \in C^k(\Omega)$ *for which* $f_j \to f$ *in the norm defined above.*

Proof. By definition of the space $H^k(\Omega)$ we know that there is a set $X_0 \subset H^k(\Omega)$ such that it is isomorphic to $C^k(\overline{\Omega})$ and X_0 is dense in $H^k(\Omega)$.

Let $\tilde{f} \in H^k(\Omega)$. Then since X_0 is dense in $H^k(\Omega)$, there is a sequence $\tilde{f}_j \in X_0$ for which

$$\lim_{j\to\infty} \left(\left\| \tilde{f} - \tilde{f}_j \right\|_{H^k} \right) = 0.$$

Let us consider the sequence $f_j \in C^k(\overline{\Omega})$ corresponding to $\tilde{f}_j$ (such sequence exists by the isomorphism). Since f_j is a Cauchy-sequence, and the norm induced by (10.1) is

$$\|f\| := \left[\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha f|^2 \right]^{1/2},$$

then for q fixed multiindex α, the sequence $(\partial^\alpha f_j)$ is a Cauchy sequence in $L^2(\Omega)$. Since $L^2(\Omega)$ is a complete space, then there is such a $f_\alpha \in L^2(\Omega)$ element such that

$$\lim_{j\to\infty} \left(\|f_\alpha - (\partial^\alpha f_j)\|_{L^2(\Omega)} \right) = 0.$$

Let f be the function f_α corresponding to the multiindex $\alpha = (0, \dots 0)$. Now we prove the following lemma.

Lemma 10.2. *The distributional derivative of the regular distribution corresponding to* f *is the regular distribution corresponding to the function* $f_\alpha \in L^2(\Omega)$, *i.e.*

$$\partial^{(\alpha)} f = f_\alpha, \qquad |\alpha| \le k.$$

Proof. (of Lemma 10.2.)

The sequence f_j tends to function f in the $L^2(\Omega)$ norm, so it also tends to it in the weak sense on the set $\mathcal{D}'(\Omega)$: for an arbitrary fixed $\varphi \in \mathcal{D}(\Omega)$ element by the Cauchy-Schwartz inequality:

$$\left| \int_\Omega f\varphi - \int_\Omega f\varphi_j \right| = \left| \int_\Omega (f - f_j)\varphi \right| \le \|f - f_j\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \to 0 \qquad \text{as } j \to \infty.$$

Then, by the definition of the weak convergence we have

$$\partial^\alpha f_j \xrightarrow{\mathcal{D}'(\Omega)} \partial^\alpha f \qquad (|\alpha| \le k).$$

Also, the sequence $(\partial^\alpha f_j)$ tends to f_α in the norm of the set$L^2(\Omega)$, so it also tends to that one in the weaker convergence of $\mathcal{D}'(\Omega)$. Therefore, distribution $\partial^\alpha f$ is the same as the regular distribution corresponding to f_α. □

(Continuation of the proof of Theorem 10.1.)

Now let us say that the element $\tilde{f} \in H^k(\Omega)$ corresponds to the function $f \in L^2(\Omega)$. It can be also shown that this element f does not depend on the choice of the approximating sequence $\tilde{f}_j$ chosen for $\tilde{f}$.

Let X_1 be the set of the functions $f \in L^2(\Omega)$ corresponding to the elements $\tilde{f} \in H^k(\Omega)$. By the previous construction this connection is a bijection, and if $f, g \in H^k(\Omega)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then the element corresponding to $\lambda_1\tilde{f} + \lambda_2\tilde{g}$ is $\lambda_1 f + \lambda_2 g$ (in which f and g are the corresponding functions to $\tilde{f}$ and $\tilde{g}$, respectively). Consequently, X_1 is a vector space (for the usual addition defined in the set $L^2(\Omega)$), and it is isomorphic to the vector space $H^k(\Omega)$.

Let us define the inner product of functions $f, g \in X_1$ by the formula (10.1), in which ∂^α means the distributional derivatives of these functions. Let $\tilde{f}$ and $\tilde{g}$ be the corresponding functions from the space H^k.

Lemma 10.3. *The following holds:*

$$\langle f, g \rangle_{X_1} = \left\langle \tilde{f}, \tilde{g} \right\rangle_{H^k}.$$

Proof. (of Lemma 10.3) Let us assume that the sequence $\tilde{f}_j$ in space X_0 tends to element $\tilde{f}$, and the sequence $\tilde{g}_j$ tends to the element $\tilde{g}$. Then by the continuity of the inner product and the previous considerations we have

$$\left\langle \tilde{f}, \tilde{g} \right\rangle_{H^k} = \lim_{j\to\infty} \left\langle \tilde{f}_j, \tilde{g}_j \right\rangle_{H^k} = \lim_{j\to\infty} \langle f_j, g_j \rangle_{H^k} = \lim_{j\to\infty} \left[\sum_{|\alpha|\le k} \int_\Omega (\partial^\alpha f_j)\,(\partial^\alpha g_j) \right] =$$

$$= \sum_{|\alpha|\le k} \int_\Omega (\partial^\alpha f)\,(\partial^\alpha g) = \langle f, g \rangle_{X_1},$$

which gives the statement. □

Consequently, the space X_1 is isomorphic to the Hilbert space $H^k(\Omega)$, which completes the proof of Theorem 10.1. □

The main problem here is that we do not want that approximation property to be in our equivalence. The next theorem solves this problem.

Definition 10.2. $\Omega \subset \mathbb{R}^n$ is said to be a **star domain**, if there is a point $x_0 \in \Omega$ ("the center of the star") that for every point $x \in \Omega$ all the points on the line starting from x_0 and passing through x are also part of Ω. See Figure 10.1.

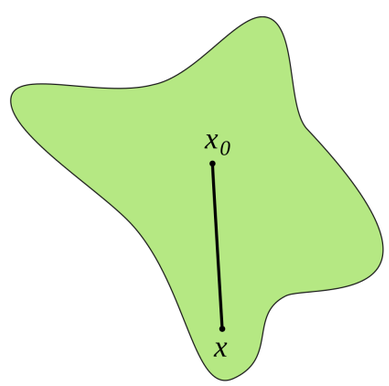


Figure 10.1: A star-domain. Here x_0 is the center of the star, and x is an arbitrary point in the domain. All the points on the line between x_0 and x are part of the domain.

Theorem 10.4. *Assume that Ω is a bounded star domain. Then the following two are equivalent.*

- $f \in H^k(\Omega)$
- *The distributional derivatives $\partial^\alpha f$, $|\alpha| \le k$ exist for $f \in L^2(\Omega)$, and these derivatives are inside $L^2(\Omega)$.*

Usually Ω is "nice", so Theorem 10.4 can be applied, so Sobolev spaces describe the space we wanted to define.

Now we define another useful space.

Definition 10.3. Consider the space $C_0^k(\overline{\Omega})$ ($k \in \mathbb{N}$). Let us define the inner product (10.1) on this space. Then the completion of the space $C_0^k(\overline{\Omega})$ with this inner product is also called a **Sobolev space**, and it is denoted by $H_0^k(\Omega)$. This is then a Hilbert space.

Then a similar theorem can be stated.

Theorem 10.5. *The following two are equivalent.*

- $f \in H_0^k(\Omega)$
- *The distributional derivatives* $\partial^\alpha f$, $|\alpha| \leq k$ *exist for* $f \in L^2(\Omega)$, *these derivatives are inside* $L^2(\Omega)$, *and there is a sequence* $f_j \in C_0^k(\Omega)$ *for which* $f_j \to f$ *in the norm defined above.*

Remark 10.3.

- If $k \geq 1$, then $H_0^k(\Omega)$ is a real subspace of $H^k(\Omega)$, meaning that there are functions which are inside $H^k(\Omega)$, but are not part of $H_0^k(\Omega)$ (the constant 1 function is a good example).
- If $k = 0$, then $H^0(\Omega) = H_0^0(\Omega) = L^2(\Omega)$.

Remark 10.4. In a more general definition of Sobolev spaces, instead of using $(.,.)_{L^2(\Omega)}$ one can also use $(.,.)_{L^p(\Omega)}$, and then we get the Sobolev space $W^{k,p}(\Omega)$: these Sobolev spaces contain functions from the space $L^p(\Omega)$ which have distributional derivatives up to order k which are inside $L^p(\Omega)$. Then it is clear that $H^k(\Omega) = W^{k,2}(\Omega)$. However, the only case when this Sobolev space is a Hilbert space is when $p = 2$, so this is the case which is used in most applications (the reason for this is that the only L^p space which is a Hilbert space is when $p = 2$).

Also, if we define Fourier transforms on the space of distributions, then one can also define the above spaces using Fourier transforms, and by generalizing that definition, one can get Sobolev spaces with fractional powers.

10.2 Basic properties of $H^k(\Omega)$ and $H_0^k(\Omega)$

Now we list some properties which are easy to prove (but their proofs are not part of the exam materials).

Proposition 10.6. *Let* Ω *be a bounded domain. If* $f \in H^k(\Omega)$ *and* $\varphi \in C^k(\overline{\Omega})$ *then* $\varphi f \in H^k(\Omega)$, *the derivatives of it can be computed using the Leibniz rule and*

$$\|\varphi f\|_{H^k} \leq c(\varphi)\|f\|_{H^k}.$$

Proposition 10.7. *If* $k < \ell$, *then* $H^\ell(\Omega) \subset H^k(\Omega)$, *and for all* $f \in H^\ell(\Omega)$ *we have*

$$\|f\|_{H^k(\Omega)} \leq \|f\|_{H^\ell(\Omega)}.$$

Proposition 10.8. *Let* $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ *be bounded sets. Then the following hold:*

- *If* $f \in H^k(\Omega_2)$, *then* $f|_{\Omega_1} \in H^k(\Omega_1)$, *and*

$$\|f\|_{H^k(\Omega_1)} \leq \|f\|_{H^k(\Omega_2)}.$$

- *If* $f \in H_0^k(\Omega_1)$, *then by extending* f *onto* Ω_2 *as a constant zero function and calling this extension* $\widetilde{f}$, *we get* $\widetilde{f} \in H_0^k(\Omega_2)$ *and*

$$\|f\|_{H_0^k(\Omega_1)} = \|\widetilde{f}\|_{H_0^k(\Omega_2)}.$$

Remark 10.5. If $f \in H_0^k(\Omega_2)$, then it might happen that $f|_{\Omega_1}$ is not part of $H_0^k(\Omega_1)$. Also, if $f \in H^k(\Omega_1)$, we do not know whether we can extend it in a way that the extension is part of $H^k(\Omega_2)$. However, if Ω_1 and Ω_2 are "nice", then there is such an extension.

10.3 Equivalent norms on $H_0^1(\Omega)$

As it was defined before, the norm on $H_0^1(\Omega)$ by definition is

$$\|f\|_{H_0^1(\Omega)} = \left[\int_\Omega \left(|f|^2 + \sum_{j=1} |\partial_j f|^2\right)\right]^{1/2}. \tag{10.5}$$

The next theorem talks about the fact that there is a more simple norm which is usually used instead of the previous one (for an application, see Chapter 11).

Theorem 10.9. *Let us assume that Ω is bounded. Then there is a constant $c_1 > 0$, for which the following holds for every $f \in H_0^1(\Omega)$:*

$$\left(\int_\Omega \sum_{j=1}^n |\partial_j f|^2\right)^{1/2} \leq \|f\|_{H_0^1(\Omega)} \leq c_1 \left(\int_\Omega \sum_{j=1}^n |\partial_j f|^2\right)^{1/2} \tag{10.6}$$

This basically means that the norm $\|f\|_{H_0^1(\Omega)}$ defined in (10.5) and the norm

$$|f|_{H_0^1(\Omega)} := \left(\int_\Omega \sum_{j=1}^n |\partial_j f|^2\right)^{1/2}$$

are equivalent.

Remark 10.6. Recap from functional analysis: norms $\|.\|$ and $|.|$ are equivalent, if for all x elements in the normed space there are some positive constants C_1 and C_2 for which

$$C_1|x| \leq \|x\| \leq C_2|x|.$$

Equivalent norms are pretty useful, since if we can prove some inequality in one of them, then the same inequality will hold in the other one also (perhaps with a different constant). A well known result is that in finite dimensional spaces all norms are equivalent (but since here we have function spaces which have infinite dimension, we cannot use this fact).

Proof. (of Theorem 10.9) We prove the statement in four steps.

1. **Step 1:** The left inequality inside (10.6) is trivial: we add up less terms (the $L^2(\Omega)$-norm of f is not on the left-hand side), so we get a smaller sum. Now we prove the right one.

2. **Step 2:** Since Ω is bounded, we can construct an n-dimensional interval (or a "rectangle") $T := [a_1, b_1] \times \cdots \times [a_n, b_n]$ for which $\Omega \subset T$ holds.

   Let us extend f onto T as $f = 0$ on $T \setminus \Omega$. Then $f \in H_0^1(T)$ and $\|f\|_{H_0^1(\Omega)} = \|f\|_{H_0^1(T)}$ and also $|f|_{H_0^1(\Omega)} = |f|_{H_0^1(T)}$. So it is enough to prove the statement on T.

3. **Step 3:** Since $C_0^1(T)$ is dense in $H_0^1(T)$, it is enough to prove the statement for all $f \in C_0^1(T)$, i.e.

$$\int_T |f|^2 \leq c_2 \int_T \sum_{j=1}^{n} |\partial_j f|^2. \tag{10.7}$$

If we can prove (10.7) then we are finished with the proof, since it is easy to see that inside $\|f\|_{H_0^1(T)}$ all the terms of $|f|_{H_0^1(T)}$ are present, so we only have to bound the one which is not there, namely $|f|^2$.

4. **Step 4: The proof of** (10.7).
Let us use the notation $x = (x_1, \tilde{x})$, in which $\tilde{x} = (x_2, x_3, \ldots, x_n)$, and then by the fundamental theorem of calculus (also know as the Newton-Leibniz formula) we have

$$f(x_1, \tilde{x}) - f(a_1, \tilde{x}) = \int_{a_1}^{x_1} \partial_1 f(y_1, \tilde{x}) dy_1.$$

However, since $f \in C_0^1(\Omega)$, we know that $f(a_1, \tilde{x}) = 0$, so:

$$|f(x_1, \tilde{x})|^2 = \left| \int_{a_1}^{x_1} \partial_1 f(y_1, \tilde{x}) dy_1 \right|^2 \leq$$

Then by using the Cauchy-Schwartz inequality with $g = 1$, we get[1]

$$\leq \int_{a_1}^{x_1} 1 \, dy_1 \int_{a_1}^{x_1} |\partial_1 f(y_1, \tilde{x})|^2 \, dy_1 =$$

$$= (x_1 - a_1) \int_{a_1}^{x_1} |\partial_1 f(y_1, \tilde{x})|^2 \, dy_1 \leq$$

Since it is an integral of a non-negative function, if the integration interval increases, then the value of the integral will not decrease, so:

$$\leq (x_1 - a_1) \int_{a_1}^{b_1} |\partial_1 f(y_1, \tilde{x})|^2 \, dy_1$$

Therefore,

$$\int_T |f|^2 dx = \int_T |f|^2 dx_1 d\tilde{x} \leq$$

$$\leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \left[(x_1 - a_1) \int_{a_1}^{b_1} |\partial_1 f(y_1, \tilde{x})|^2 dy_1 \right] dx_n \ldots dx_2 dx_1 =$$

Note that the term $|\partial_1 f(y_1, \tilde{x})|^2$ does not depend on x_1, and $(x_1 - a_1)$ does not depend on any of the other variables. Because of this, these two can be integrated separately:

$$= \left(\prod_{i=2}^{n} (b_i - a_i) \right) \left(\int_{a_1}^{b_1} (x_1 - a_1) dx_1 \right) \cdot$$

$$\cdot (b_1 - a_1) \left(\int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \left[\int_{a_1}^{b_1} |\partial_1 f(y_1, \tilde{x})|^2 dy_1 \right] dx_n \ldots dx_2 \right) =$$

[1] The Cauchy-Schwartz inequality says that $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$. Here $f = \partial_1 f(y_1, \tilde{x})$, $g = 1$ and the norm and the inner product are the usual ones defined in L^2, namely $\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega fg$ and $\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f|^2 \right)^{1/2}$.

$$= \left(\prod_{i=1}^{n} (b_i - a_i) \right) \frac{(b_1 - a_1)^2}{2} \int_T |\partial_1 f|^2 dx$$

Note that

$$\prod_{i=1}^{n} (b_i - a_i) = \lambda(T)$$

where $\lambda(T)$ is the Lebesgue-measure (or volume) of rectangle T.

Since the previous argument can be repeated for every variable, we get that

$$\int_T |f|^2 dx \leq \lambda(T) \frac{(b_j - a_j)^2}{2} \int_T |\partial_j f|^2$$

for any index j, $1 \leq j \leq n$. Then if we add up all of these inequalities, we get

$$n \int_T |f|^2 dx \leq \sum_{j=1}^{n} \lambda(T) \frac{(b_j - a_j)^2}{2} \int_T |\partial_j f|^2.$$

From which we get

$$\int_T |f|^2 dx \leq c_2 \int_T \sum_{j=1}^{n} |\partial_j f|^2$$

with $c_2 = \lambda(T) \max_j \frac{(b_j - a_j)^2}{2}$, which is (10.7), so we are done.

Consequently, the statement of the theorem is proved. □

Remark 10.7. The inequality inside (10.6) is not true in the case of $H^1(\Omega)$: for example for $f \equiv 1$ it does not hold (since here $\|f\|_{H_0^1(\Omega)} = \lambda(\Omega)$ which is the Lebesgue-measure of Ω, but $|f|_{H_0^1(\Omega)} = 0$). This also shows that $H_0^1(\Omega)$ is a real subspace of $H^1(\Omega)$.

Remark 10.8. If Ω is not bounded, then the inequality is not necessarily true. However, it can be proved that it holds for $\Omega = \mathbb{R}^n$.

10.4 The trace operator

The main question of this section is the way the boundary values of a function inside $H^k(\Omega)$ can be defined. In the case of continuous functions the boundary values are well-defined - however, in L^p-spaces functions are defined only almost everywhere[2], so since the boundary is usually a set with zero measure, we cannot really define the values of such functions on it. Fortunately, Sobolev spaces are defined in a way that their restriction to a domain with zero measure makes sense: then the trace operator will solve this problem.

Theorem 10.10. *Let $\Omega \subset \mathbb{R}^n$ be bounded and "nice"[3]. Then for any $f \in C^1(\overline{\Omega})$ function, the map*

$$f \to f|_{\partial\Omega}$$

is continuous as a linear operator acting between $H^1(\Omega) \to L^2(\partial\Omega)$.

[2]It means that there is a set with zero measure on which the function might take any arbitrary (real) values.

[3]We should assume that Ω is locally brick-like, meaning that there is a diffeomorphism between every neighborhood of a point in Ω and a neighborhood of a point inside a "brick", an n-dimensional interval or rectangle.

Corollary 10.11. *Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, then the map $Lf := f|_{\partial\Omega}$ defined for $f \in C^1(\overline{\Omega})$, which is linear and bounded, can be extended uniquely to $H^1(\Omega)$.*

In the previous corollary we used the fact that any bounded linear operator defined on a dense subset of a normed space can be extended uniquely to the whole space.

Definition 10.4. The above extension of the operator L,

$$\widetilde{L} : H^1(\Omega) \to L^2(\partial\Omega)$$

is a bounded and linear operator called the **trace operator**, and $\widetilde{L}f$ is called the **trace** of $f \in H^1(\Omega)$ on $\partial\Omega$.

Theorem 10.12. *Let $\Omega \subset \mathbb{R}^n$ be bounded and "nice"[4]. Then $\tilde{L} : H^1(\Omega) \to L^2(\partial\Omega)$ is a compact operator.*

Remark 10.9. Small recap from functional analysis: an operator is compact, if a map of a closed set is relatively compact (meaning that its closure is compact).

Theorem 10.13. *Let $\Omega \subset \mathbb{R}^n$ be bounded and "nice"[5]. Then the following two are equivalent:*

1. *$f \in H_0^1(\Omega)$*
2. *$f \in H^1(\Omega)$, and $f|_{\partial\Omega} = 0$ (in the trace sense).*

Proof.

1. Case 1. $\Rightarrow$ 2.
   If $f \in H_0^1(\Omega)$, then there is a sequence $f_j \in C_0^1(\Omega)$ for which $f_j \to f$ in the H^1-norm. But we know that $f_j|_{\partial\Omega} = 0$, so by the convergence we get $f|_{\partial\Omega} = 0$.
2. Case 2. $\Rightarrow$ 1.
   This is rather long and technical, so it is skipped.

□

So basically this means that the functions inside $H_0^1(\Omega)$ are those functions in $H^1(\Omega)$ which are zero on the boundary (in the "trace sense" defined above).

10.5 The embedding of $H_0^1(\Omega)$ and $H^1(\Omega)$ into $L^2(\Omega)$ (skipped in 2024)

This section talks about the embedding of Sobolev spaces into L^2-spaces, which will play an important role in the next chapter.

Theorem 10.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the embedding of $H^1(\Omega)$ and $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, i.e. $H_0^1(\Omega) \subset L^2(\Omega)$ (or $H^1(\Omega) \subset L^2(\Omega)$), and if $\widetilde{f} \in H_0^1(\Omega)$ (or $\widetilde{f} \in H^1(\Omega)$), then there is an $\widetilde{f} \to f \in L^2(\Omega)$ compact operator.*

Remark 10.10. The compact property can also be expressed in the following way:
If f_k is a bounded sequence in $H_0^1(\Omega)$ (or in $H^1(\Omega)$), then there is a sub-sequence which is convergent in the L^2-norm.

In the next chapter we state theorems about the existence of solutions of the general boundary value problems.

[4]We should assume that Ω is locally brick-like, see the previous footnote
[5]We should assume that Ω is locally brick-like, see the previous footnote

Chapter 11

Weak form of boundary value problems

In this chapter we define the weak form of boundary-value problems, more precisely the weak form of the first boundary-value problem.

As it was defined before, the **classical first boundary-value problem** is the following. Consider an $\Omega \subset \mathbb{R}^n$ bounded domain, $p \in C^1(\overline{\Omega})$, $p(x) \geq m > 0$, $q, f \in C(\overline{\Omega})$, $q > 0$, $g \in C(\partial\Omega)$. Then we search for a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for which the following holds:

$$\begin{cases} -\mathrm{div}(p\,\mathrm{grad}(u)) + qu = f & \text{inside } \Omega, \\ u|_{\partial\Omega} = g. \end{cases} \tag{11.1}$$

In Chapter 8 we proved that this problem has only one solution, but the existence of such solutions is still an open question. In this chapter we will prove this latter one.

(Note that in Chapter 8 the first boundary value problem had also a given function inside the boundary condition, which is now assumed to be constant one.)

Our goal is now to prove the existence of solutions in the weak sense, and then by a connection between the weak and classical solutions, we will get the desired statement.

11.1 Weak form of boundary-value problems

As it was mentioned before, the **weak form** of an equation means that we do not take the derivatives of the functions in the classical (i.e. "usual derivative") sense, but we consider the distributional derivatives in the equation. This latter one means the following: if we have a "nice" function f (i.e. it is in L^1_{loc}), then we can define a regular distribution corresponding to it, namely T_f. Then we can compute the derivative of this distribution, and if this derivative is also a regular distribution, i.e. $\partial T_f = T_g$, then we say that the weak derivative of f is g. Or, in other words, if we take the derivative only in one variable, then:

$$\partial_i T_f = \partial_i \left(\int_\Omega f\varphi \right) = -\int_\Omega f \partial_i \varphi = \int_\Omega g\varphi,$$

so if $\displaystyle\int_\Omega f\partial_i\varphi = -\int_\Omega g\varphi$, then we say that $\partial_i f = g$ in the distributional sense. Note that the main gain of this whole process is that here f should not be

differentiable, and it is enough that the integral of $f\partial_i\varphi$ exists, which is true if $f \in L^2$ (this was the space on which the Sobolev spaces were defined).

Now the main question is how to get such an $\int_\Omega g\varphi$ term in our equation. The main idea here is to multiply our equation by φ, and then integrate it on Ω.

Let us suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution of the first boundary-value problem (11.1). Let us multiply the first equation with $v \in C_0^1(\overline{\Omega})$ and integrate it on Ω afterwards! Then we get the following form:

$$\int_\Omega -\mathrm{div}(p\ \mathrm{grad}(u))v + quv = \int_\Omega fv. \tag{11.2}$$

Now let us use the first Green theorem, namely

$$\int_\Omega -\mathrm{div}(p\ \mathrm{grad}(u))\ v = \int_\Omega p\ \mathrm{grad}u \cdot \mathrm{grad}v - \int_{\partial\Omega} p\ v\ \partial_\nu u\ d\sigma.$$

By definition, $v = 0$ on the boundary, so the second term is zero in our case. Then equation (11.2) can be rewritten as

$$\int_\Omega p\ \mathrm{grad}u \cdot \mathrm{grad}v + \int_\Omega quv = \int_\Omega fv. \tag{11.3}$$

Equation (11.3) is called **the weak form** of the boundary-value problem (11.1).

Note that since equation (11.3) holds for every $v \in C_0^1(\overline{\Omega})$, it also holds for every $v \in H_0^1(\overline{\Omega})$ (since we can construct a sequence of functions inside $C_0^1(\overline{\Omega})$ which tend to the element of $H_0^1(\overline{\Omega})$). Also, in this case $u \in H^1(\Omega)$, and $u|_{\partial\Omega} = g$ in the trace sense.

Definition 11.1. The **general (or weak) solution** of the first boundary-value problem is such a $u \in H^1(\Omega)$ function, for which (11.3) holds $\forall v \in H_0^1(\Omega)$ and $u|_{\partial\Omega} = g$ in the trace sense.

By the previous arguments we showed that if u is a classical solution of the first boundary-value problem, then u is also a weak solution of this problem, so in this sense it is a generalization of the classical problem. The next theorem sum up the connections between these two terms.

Theorem 11.1 (Connection between the classical and weak solutions)**.**

1. *If u is a classical solution of the first boundary problem and $u \in C^1(\overline{\Omega})$, then u is also a weak solution of the same problem.*
2. *If $u \in H^1(\Omega)$ is a weak solution, and also $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then u is also a classical solution.*

Proof.

1. See the previous arguments.
2. If (11.3) holds, and also $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then the first Green theorem can be applied to (11.3) in the other direction, so we get:

$$\int_\Omega v\left(-\mathrm{div}(p\ \mathrm{grad}(u)) + qu - f\right) = 0 \tag{11.4}$$

Then (11.4) holds for every $v \in C_0^1(\Omega)$, and since $C_0^1(\Omega)$ is dense in $L^2(\Omega)$, we get that

$$-\text{div}(p\,\text{grad}(u)) + qu - f = 0,$$

which is the equation we wanted.

Thus, the statement is proved. □

This theorem states that if we have a general solution and it is smooth enough, then it is going to be a classical one, meaning that if we can prove that the weak problem has a unique solution and it is smooth enough, then it is also a unique solution of the classical problem.

11.2 Existence of solutions

The existence of solutions of problem (11.3) can be proved in several ways, but most of these involve some theorems from functional analysis (e.g. Lax-Milgram lemma or other fixed point theorems). Here we are going to use the Riesz representation theorem.

From now on, we will assume that $g = 0$, meaning that we have a homogeneous Dirichlet condition (the next theorems can also be proved in the case of $g \neq 0$).

Let us define an inner product in the space $H_0^1(\Omega)$ in the following way:

$$\{u, v\} := \int_\Omega p\,\text{grad}u \cdot \text{grad}v + quv \tag{11.5}$$

It can be proved that it is indeed an inner product.

Proposition 11.2. *Inner product* (11.5) *generates a norm*[1] *which is equivalent to the usual* $H_0^1(\Omega)$ *norm.*

Proof. The square of the norm generated by (11.5) is

$$\{u, u\} := \int_\Omega p\,\text{grad}u \cdot \text{grad}u + \int_\Omega qu^2. \tag{11.6}$$

We have to bound this expression form above and from below using the norm of $H_0^1(\Omega)$.

The bound from below:

$$\int_\Omega p\,\text{grad}u \cdot \text{grad}u + \int_\Omega qu^2 \geq$$

Now we use that $q \geq 0$, so the second term is non-negative, so:

$$\geq \int_\Omega p\,\text{grad}u \cdot \text{grad}u \geq$$

Now we use the fact that $p \geq m > 0$, so

$$\geq \int_\Omega m\,|\partial_i u|^2 \geq c_2 \left(\|u\|'_{H_0^1(\Omega)}\right)^2$$

[1] An inner product (u, v) generates a norm with the formula $\|u\| = \sqrt{(u, u)}$.

in which $\|.\|'_{H_0^1(\Omega)}$ is the norm defined in the Section 10.3, which was proved to be equivalent to the usual Sobolev norm.

The other direction is also similar:

$$\int_\Omega p\,\mathrm{grad}u\cdot\mathrm{grad}u+\int_\Omega qu^2\leq$$

Now by using the Cauchy-Schwartz inequality (with $g=1$):

$$\leq \text{const}\int_\Omega\left[|\partial u|^2+|u|^2\right]=\text{const}\,\|u\|^2_{H_0^1(\Omega)},$$

which means that these two norms are equivalent. □

If we use the inner product defined above, our equation (11.3) can be rewritten as

$$\{u,v\}=\int_\Omega fv$$

Theorem 11.3. *Problem* (11.3) *has a unique solution.*

Proof. Let us define operator B in the following way:

$$B(v):=\int_\Omega fv,\qquad v\in H_0^1(\Omega)$$

It is clear that B is a linear functional. Also, B is bounded, since

$$|B(v)|=\left|\int_\Omega fv\right|\leq$$

By applying the Cauchy-Schwartz inequality:

$$\leq\|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\leq$$

By Proposition 11.2 (and by the proof of Theorem 10.9):

$$\leq\text{const}\,\|f\|_{L^2(\Omega)}\sqrt{\{v,v\}}.$$

Then if we consider the inner product space $H_0^1(\Omega)$ equipped with the inner product $\{.,.\}$, then in this space B is a linear, bounded functional. This means that we can apply the Riesz representation theorem, which is the following:

Theorem 11.4 (Riesz representation theorem)**.** *If H is a Hilbert space with inner product $\langle.,.\rangle$ and φ is a linear, bounded functional, then there is a unique element $u\in H$ for which*

$$\langle u,x\rangle=\varphi(x)\qquad\forall x\in H.$$

(Its proof can be found in any good functional analysis textbook.)

Then we get that there exists a unique element $u\in H_0^1(\Omega)$ for which

$$\{u,v\}=B(v),\qquad\forall v\in H_0^1(\Omega),$$

which by definition means that there is a unique element $u\in H_0^1(\Omega)$ for which

$$\int_\Omega p\,\mathrm{grad}u\cdot\mathrm{grad}v+\int_\Omega quv=\int_\Omega fv$$

so we got that the weak problem has a unique solution. □

The next corollary is a consequence of Theorems 11.1. and 11.3., which is also the main result of this chapter.

Corollary 11.5. *The classical (homogeneous) first boundary-value problem has a unique solution.*

Remark 11.1. Similar arguments also hold for time-dependent (mixed) problems, in which there are also some initial conditions next to the boundary conditions.

Remark 11.2. In most applications, the weak forms of PDEs are considered, since one of the most effective numerical method, *the finite elements method* is designed to solve weak problems, so the above formulation is widely used in applications.