1, a, (4) $-3+3i = 3\sqrt{2}\, e^{i\frac{3\pi}{4}}$ (2) ; $\ln(-3+3i) = \underline{\underline{\ln(3\sqrt{2}) + i\frac{3\pi}{4}}}$ (2)

b, (6) $\dfrac{-4+6i}{1+5i} = \dfrac{(-4+6i)(1-5i)}{1^2+5^2} = \dfrac{-4+30+20i+6i}{26} = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$ (3)

$$\left(\frac{-4+6i}{1+5i}\right)^{2019} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{2019} = 2^{1009}\sqrt{2}\cdot e^{i\pi\left(504+\frac{3}{4}\right)} = 2^{1009}\sqrt{2}\cdot e^{i\frac{3\pi}{4}} =$$

$$= 2^{1009}\sqrt{2}\cdot\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \underline{\underline{2^{1009}(-1+i)}} \quad (3)$$

2, T.: $\underline{\underline{A}} \in \mathrm{Lin}\, \mathbb{V}$ diagonalizálható $\iff$ $\mathbb{V}$-nek van $\underline{\underline{A}}$ sajátvektoraiból álló bázisa.
(3)

B.: (7) $(\Longrightarrow)$ Ha $[\underline{\underline{A}}]^{\mathcal{B}} = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ diagonális a $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \dots \underline{b}_n\}$ bázisban, akkor $\underline{\underline{A}}\,\underline{b}_k = d_k\,\underline{b}_k$, tehát $\forall\, \underline{b}_k$ sajátvektor. ✓

$(\Longleftarrow)$ Ha $\mathcal{B} = \{\underline{b}_1, \dots \underline{b}_n\} \subset \mathbb{V}$ bázis, és $\underline{\underline{A}}\,\underline{b}_k = \lambda_k\,\underline{b}_k$, akkor

$[\underline{\underline{A}}]^{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ diagonális ✓

(Mindkét esetben azt használtuk ki, hogy $\underline{\underline{A}}\,\underline{b}_k$ kifejtési együtthatói állnak a $[\underline{\underline{A}}]^{\mathcal{B}}$ mátrix k. oszlopában.)

3, a, $f, g \in \mathcal{P}$, $\alpha, \beta \in \mathbb{R}$ esetén

(3) $(A(\alpha f + \beta g))(x) = (x-1)(\alpha f + \beta g)'(x) = (x-1)\left(\alpha f'(x) + \beta g'(x)\right) =$

$= \alpha(x-1) f'(x) + \beta(x-1) g'(x) = \alpha(Af)(x) + \beta(Ag)(x) = \left(\alpha A f + \beta A g\right)(x)$ $\quad\forall x$-re

$\Longrightarrow A(\alpha f + \beta g) = \alpha A f + \beta A g$

3, b, $\boxed{7}$

$$\left.\begin{aligned}
(A b_0)(x) &= (x-1)\, b_0'(x) = 0 \\
(A b_1)(x) &= (x-1)\, b_1'(x) = x-1 = (b_1 - b_0)(x) \\
(A b_2)(x) &= (x-1)\, b_2'(x) = (x-1)2x = (2b_2 - 2b_1)(x) \\
(A b_3)(x) &= (x-1)\, b_3'(x) = (x-1)3x^2 = (3b_3 - 3b_2)(x)
\end{aligned}\right\} [A] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & +2 & -3 \\ 0 & 0 & 0 & +3 \end{bmatrix}$$

([A] oszlopai a kifejtési együtthatók.)

4, a, $p = 1$.

$\boxed{7}$

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -3 \\ 1 & 1 & 1 \\ 0 & -2 & -3 \end{vmatrix} = -1 \underbrace{\begin{vmatrix} 1 & -3 \\ -2 & -3 \end{vmatrix}}_{-3-6} = +9$$

$$D_x = \begin{vmatrix} -5 & 3 & -1 \\ 6 & 1 & 1 \\ -5 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 0 \end{vmatrix} = -1 \underbrace{\begin{vmatrix} 1 & 4 \\ 7 & 1 \end{vmatrix}}_{1-28} = +27\ ; \quad x = \frac{D_x}{D} = \underline{\underline{3}}$$

$$D_y = \begin{vmatrix} 2 & -5 & -1 \\ 1 & 6 & 1 \\ 1 & -5 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 3 \\ 0 & 11 & 3 \\ 1 & -5 & -2 \end{vmatrix} = +1 \underbrace{\begin{vmatrix} 5 & 3 \\ 11 & 3 \end{vmatrix}}_{15-33=-18} = -18\ ; \quad y = \frac{D_y}{D} = \underline{\underline{-2}}$$

$$D_z = \begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & 6 \\ 1 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 5 \\ 0 & 2 & 11 \\ 1 & -1 & -5 \end{vmatrix} = +1 \underbrace{\begin{vmatrix} 5 & 5 \\ 2 & 11 \end{vmatrix}}_{55-10} = 45\ ; \quad z = \frac{D_z}{D} = \underline{\underline{5}}$$

b, $\boxed{3}$

$$D(p) = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & p \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 3 \\ 0 & 2 & p+2 \\ 1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 2 & p+2 \end{vmatrix} = 5(p+2) - 6 =$$

$= 5p + 4 \overset{!}{=} 0 \implies \underline{\underline{p = -\frac{4}{5}}}$ esetén nincs egyértelmű megoldás.

5*, a, D: $\underline{\underline{U}} \in \mathrm{Lin}\, \mathbb{V}$ unitér, ha $\exists\, \underline{\underline{U}}^{-1} = \underline{\underline{U}}^{*}$ (inverze megegyezik az adjungáltjával)
$\boxed{2}$

b, T.: i, Ha $\underline{\underline{U}}$ unitér, és λ sajátérték, akkor $|\lambda| = 1$. (1)

ii, Ha $\underline{\underline{U}}$ unitér, és $\underline{u}$ és $\underline{v}$ két különböző sajátértékhez tartozó sajátvektor, akkor $\underline{u} \perp \underline{v}$. (1)

B.: i, $\underline{\underline{U}}^{-1} = \underline{\underline{U}}^{*}$, $\underline{\underline{U}}\,\underline{v} = \lambda \underline{v}$ esetén

$$\left.\begin{aligned} &\langle \underline{v}, \underline{v} \rangle = \langle \underline{v}, \underbrace{\underline{\underline{U}}^{-1}\underline{\underline{U}}}_{\underline{\underline{I}}}\, \underline{v} \rangle = \langle \underline{\underline{U}}\,\underline{v}, \underline{\underline{U}}\,\underline{v} \rangle = \langle \lambda \underline{v}, \lambda \underline{v} \rangle = |\lambda|^2 \langle \underline{v}, \underline{v} \rangle \\ &\qquad\qquad\qquad \Rightarrow |\lambda|^2 = 1 \ \checkmark \end{aligned}\right\} (3)$$

(3)

ii, $\underline{\underline{U}}^{-1} = \underline{\underline{U}}^{*}$, $\underline{\underline{U}}\,\underline{v} = \lambda \underline{v}$, $\underline{\underline{U}}\,\underline{u} = \mu \underline{u}$, $\lambda \neq \mu$ esetén:

$$\langle \underline{v}, \underline{u} \rangle = \langle \underline{v}, \underline{\underline{U}}^{-1}\underline{\underline{U}}\, \underline{u} \rangle = \langle \underline{\underline{U}}\,\underline{v}, \underline{\underline{U}}\,\underline{u} \rangle = \langle \lambda \underline{v}, \mu \underline{u} \rangle = \bar{\lambda}\mu \langle \underline{v}, \underline{u} \rangle$$

Tehát $\langle \underline{v}, \underline{u} \rangle (1 - \bar{\lambda}\mu) = 0$.

$1 - \bar{\lambda}\mu \neq 0$, hiszen $|\lambda| = |\mu| = 1$, tehát $\bar{\lambda} = \frac{1}{\lambda}$, $1 - \bar{\lambda}\mu = 0 \Leftrightarrow \frac{\mu}{\lambda} = 1 \Leftrightarrow \mu = \lambda$

Tehát $\langle \underline{v}, \underline{u} \rangle = 0$ ✓

6*, a, $\boxed{5}$

$$\begin{vmatrix} -2-\lambda & -5 & -4 \\ -2 & 4-\lambda & 5 \\ 8 & 2 & -2-\lambda \end{vmatrix} = \underbrace{(2+\lambda)^2}_{\lambda^2+4\lambda+4}(4-\lambda) - 200 + 16 - \left(-32(4-\lambda) - 10(2+\lambda) - 10(2+\lambda)\right) =$$

$$= -\lambda^3 + \cancel{12\lambda} + \cancel{16} - \cancel{200} + \cancel{16} + \cancel{128} - \cancel{32\lambda} + \cancel{40} + \cancel{20\lambda} = -\lambda^3 = 0 \Rightarrow \lambda = 0 \quad (2)$$

$$\left[\begin{array}{ccc|c} -2 & -5 & -4 & 0 \\ -2 & 4 & 5 & 0 \\ 8 & 2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 5/2 & 2 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & -18 & -18 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 5/2 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} &z \in \mathbb{R} \\ &y = -z \\ &x = -\tfrac{5}{2}y - 2z = \tfrac{z}{2} \end{aligned}$$

Tehát a $\lambda = 0$-hoz tartozó sajátaltér: $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \mathbb{R}$ (3)

b, A nilpotens, (1) $A^n = 0 \ \forall n \geq 3$ esetén

c, $\boxed{4}$ $\cos A = \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n}}{(2n)!} = I - \frac{A^2}{2} \overset{(2)}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -2 & -5 & -4 \\ -2 & 4 & 5 \\ 8 & 2 & -2 \end{bmatrix} \begin{bmatrix} -2 & -5 & -4 \\ -2 & 4 & 5 \\ 8 & 2 & -2 \end{bmatrix} =$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -18 & -18 & -9 \\ +36 & +36 & 18 \\ -36 & -36 & -18 \end{bmatrix} = \begin{bmatrix} 10 & 9 & 9/2 \\ -18 & -17 & -9 \\ 18 & 18 & 10 \end{bmatrix} \quad (2)$$