Problems for practicing

1. Show that if every subsequence of (x_n) has a convergent subsequence converging to a, then $x_n \to a$.

2. Determine the limit of the following sequence.

$$(x_n, y_n) = \left(\sqrt[n]{n^2(2^n+5)}, \left(\frac{2n-3}{2n+4}\right)^{n+1} \right)$$

3. Show that $\partial(A \cup B) \subset \partial A \cup \partial B$ and $\partial(A \cap B) \subset \partial A \cup \partial B$ hold for every $A, B \subset \mathbb{R}^p$.

4. Prove that the boundary of every subset of $\mathbb{R}^p$ is closed.

5. Show that for any $A \subset \mathbb{R}^p$ we have $\partial(\operatorname{cl} A) \subset \partial A$, where $\operatorname{cl} A$ denotes the closure of A. Give an example for strict and for non-strict inclusion.

6. Determine the limit of the following functions at the origin. (Prove if they do not exist.)

$$\frac{xy^2}{x^2+y^4}; \qquad \frac{x^5+y^4}{x^2+y^4}; \qquad \frac{x^5+y^5}{x^2+y^4}; \qquad \frac{x^2}{x+y}$$

7. $$f(x,y) = \begin{cases} \dfrac{y^p}{x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

   For $p = 2, 3, 4$ give the domain and the value of the functions f'_x, f'_y and $\operatorname{grad} f$. Calculate the directional derivatives of f (for $p = 2, 3, 4$) at the origin.

8. Let $f(x,y) = g\left(\frac{x}{1+y^2}\right)$, where $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Determine the second order derivative (Hesse matrix) of f.

9. Determine the (local) extremal points of $f(x,y) = (x-y+1)^2 - (x^2-2)^2$.

10. Determine the global extremal points and extremal values of the function $f(x,y) = x^3y^5$ on the closed triangle with vertices $A = (0,0)$, $B = (1,0)$, $C = (0,1)$.