Exercise-book
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Part-IV
Two-dimensional continuous distributions
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Contents

1 Two-dimensional random variables and distributions

PROBLEMS

1. *Calculating probabilities by summation, using Excel*
   The distribution of a two-dimensional random variable (X, Y) is given by:

5	0.002	0.013	0.039	0.058	0.044	0.013
4	0.004	0.028	0.083	0.124	0.093	0.028
3	0.003	0.024	0.071	0.107	0.080	0.024
2	0.001	0.010	0.03	0.046	0.034	0.01
1	0.000	0.002	0.007	0.010	0.007	0.002
0	0.000	0.000	0.001	0.001	0.001	0.000
y / x	**0**	**1**	**2**	**3**	**4**	**5**

   Calculate the probability that $6 < X + Y < 9$.

   Solution
   Sol-04-01-01

2. *Continuation of the previous problem*
   Invent other events related to the two-dimensional random variable (X, Y), and calculate their probabilities by summation.

3. *A two dimensional discrete random variable*
   Simulate with Excel

   (a) a random variable Y uniformly distributed on the set $1, 2, ...n$, where n is a parameter;

   (b) a random variable X uniformly distributed on the set $1, 2, 3, 4, 5, 6$, and then a random variable Y uniformly distributed on the set $1, 2, ..., X$, and then consider the two-dimensional random variable (X, Y). What is the distribution of (X, Y)?

4. *Working with two-dimensional density functions*
   For each of the cases below, the density function $f(x, y)$ of a two-dimensional random point (X, Y) is defined. In each case, determine the density function $f_1(x)$ of X and the density function $f_2(x)$ of Y. Try to visualize the densities by mass distributions. Try to draw the graphs of the densities (a surface for $f(x, y)$, a curve for $f_1(x)$ and for $f_2(y)$). Imagine that, using a computer, we generate a point-cloud for (X, Y) and project it onto the x- and y-axes to get the point-clouds for X and Y. Try to draw (with your pencil or pen) the point-clouds which you think would resemble to the point-clouds made by the computer.

   (a) $f(x, y) = 1 \quad (0 < x < 1, \;\; 0 < y < 1)$;

(b) $f(x,y) = 1/6 \quad (0 < x < 3, \;\; 0 < y < 2)$;
(c) $f(x,y) = 4xy \quad (0 < x < 1, \;\; 0 < y < 1)$;
(d) $f(x,y) = 6xy^2 \quad (0 < x < 1, \;\; 0 < y < 1$;
(e) $f(x,y) = e^{-x-y} \quad (x > 0, \;\; y > 0)$;
(f) $f(x,y) = 10e^{-5x-2y} \quad (x > 0, \;\; y > 0$;
(g) $f(x,y) = 2 \quad (x > 0, \;\; y > 0, \;\; 0 < x + y < 1)$;
(h) $f(x,y) = 2 \quad (0 < x < y < 1)$;
(i) $f(x,y) = 6(y - x) \quad (0 < x < y < 1)$;
(j) $f(x,y) = 3x \quad (0 < x < y < 1)$;
(k) $f(x,y) = 3(1 - y) \quad (0 < x < y < 1)$.

Moreover, calculate the probabilities:

(a) $P\,(X < 0.5)$;
(b) $P\,(X + Y < 1)$;
(c) $P\,(Y > X + 0.5)$;
(d) $P\left(Y > \sqrt{X}\right)$.

2 Uniform distribution on a two-dimensional set

PROBLEMS

5. *Uniform distribution on a square*
Assume that (X, Y) follows uniform distribution on the unit square $\{(x, y) : 0 < x < 1, 0 < y < 1\}$. Calculate the probabilities and conditional probabilities:
   (a) $P\,(X < 0.8)$;
   (b) $P\,(Y > 0.3)$;
   (c) $P\,(X < 0.8 \text{ and } Y > 0.3)$;
   (d) $P\,(X < 0.8 \mid Y > 0.3)$;
   (e) $P\,(Y > 0.3 \mid X < 0.8)$;
   (f) $P\,(X + Y < 0.8)$;
   (g) $P\,(Y < X/2)$;
   (h) $P\left(Y < X^2\right)$.

6. *Uniform distribution on a triangle*
A random point (X, Y), uniformly distributed in the triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 3)$, is considered. Calculate the probabilities:
   (a) $P(Y < X)$;
   (b) $P(Y < X^2)$.
   (c) Determine both the distribution and the density function of Y/X.

3 *** Beta distributions in two-dimensions

EXCEL

The following files show two-dimensional beta point-clouds. Study them.

Demonstration file: Two-dimensional beta point-cloud related to size 2 and ranks 1 and 2
ef-200-69-00

Demonstration file: Two-dimensional beta point-cloud related to size 3 and ranks 1 and 2
ef-200-70-00

Demonstration file: Two-dimensional beta point-cloud related to size 3 and ranks 1 and 3
ef-200-71-00

Demonstration file: Two-dimensional beta point-cloud related to size 3 and ranks 2 and 3
ef-200-72-00

Demonstration file: Two-dimensional beta point-cloud related to size 5 and arbitrary ranks
ef-200-73-00

Demonstration file: Two-dimensional beta point-cloud related to size 10 and arbitrary ranks
ef-200-74-00

The following file serves to study two-dimensional point-clouds for arrival times.

Demonstration file: Two-dimensional gamma distribution
ef-200-68-00

PROBLEMS

7. *Smallest and biggest of 10 random numbers*
Three independent random numbers (uniformly distributed between 0 and 1) are generated. Let X be the smallest, Y be the biggest of them. Determine the density function $f(x, y)$ of (X, Y), visualize the density by a mass distribution, try to draw the graph (a surface) for $f(x, y)$, and if you have access to a computer, then generate a point-cloud for (X, Y) Moreover, calculate the probabilities:

   (a) $P(X < 0.5)$;

   (b) $P(X + Y < 1)$;

   (c) $P(Y > X + 0.5)$;

   (d) $P\left(Y > \sqrt{X}\right)$.

8. *Smallest and second smallest of 3 random numbers*
Three independent random numbers (uniformly distributed between 0 and 1) are generated. Let X be the smallest, Y be the second smallest of them. Determine the density function $f(x, y)$ of (X, Y), visualize the density by a mass distribution, try to draw the graph (a surface) for $f(x, y)$, and if you have access to a computer, then generate a point-cloud for (X, Y) Moreover, calculate the probabilities:

   (a) $P(X < 0.5)$;

   (b) $P(X + Y < 1)$;

   (c) $P(Y > X + 0.5)$;

   (d) $P\left(Y > \sqrt{X}\right)$.

9. *Second smallest and biggest of 3 random numbers*
Three independent random numbers (uniformly distributed between 0 and 1) are generated. Let X the second smallest, Y the biggest of them. Determine the density function $f(x, y)$ of (X, Y), visualize the density by a mass distribution, try to draw the graph (a surface) for $f(x, y)$, and if you have access to a computer, then generate a point-cloud for (X, Y) Moreover, calculate the probabilities:

   (a) $P(X < 0.5)$;

   (b) $P(X + Y < 1)$;

   (c) $P(Y > X + 0.5)$;

   (d) $P\left(Y > \sqrt{X}\right)$.

10. *Smallest and second smallest of 4 random numbers*
Four independent random numbers (uniformly distributed between 0 and 1) are generated. Let X be the smallest, Y be the biggest of them. Determine the density function $f(x, y)$ of (X, Y), visualize the density by a mass distribution, try to draw the graph (a surface) for $f(x, y)$, and if you have access to a computer, then generate a point-cloud for (X, Y) Moreover, calculate the probabilities:

(a) $P(X < 0.5)$;
(b) $P(X + Y < 1)$;
(c) $P(Y > X + 0.5)$;
(d) $P\left(Y > \sqrt{X}\right)$.

11. *Smallest and second smallest of 4 random numbers*
Four independent random numbers (uniformly distributed between 0 and 1) are generated. Let X = the smallest, Y = the second smallest of them. Determine the density function $f(x, y)$ of (X, Y), visualize the density by a mass distribution, try to draw the graph (a surface) for $f(x, y)$, and if you have access to a computer, then generate a point-cloud for (X, Y) Moreover, calculate the probabilities:

    (a) $P(X < 0.5)$;
    (b) $P(X + Y < 1)$;
    (c) $P(Y > X + 0.5)$;
    (d) $P\left(Y > \sqrt{X}\right)$.

12. *Second smallest and third smallest of 4 random numbers*
Four independent random numbers (uniformly distributed between 0 and 1) are generated. Let X be the second smallest, Y be the third smallest of them. Determine the density function $f(x, y)$ of (X, Y), visualize the density by a mass distribution, try to draw the graph (a surface) for $f(x, y)$, and if you have access to a computer, then generate a point-cloud for (X, Y) Moreover, calculate the probabilities:

    (a) $P(X < 0.5)$;
    (b) $P(X + Y < 1)$;
    (c) $P(Y > X + 0.5)$;
    (d) $P\left(Y > \sqrt{X}\right)$.

4 Projections and conditional distributions

EXCEL

Here are some files to visualize projections and conditional distributions. Study them.

Demonstration file: $X = RND_1$, $Y = XRND_2$, projections and conditional distributions
ef-200-79-00

Demonstration file: Two-dim beta distributions, $n = 10$, projections and conditional distributions
ef-200-80-00

Demonstration file: Two-dim beta distributions, $n \leq 10$, projections and conditional distributions
ef-200-81-00

Here are some files to study construction of a two-dimensional continuous distribution using conditional distributions:

Demonstration file: Conditional distributions, uniform on parallelogram
ef-200-84-00

Demonstration file: Conditional distributions, (RND1;RND1RND2)
ef-200-85-00

Demonstration file: Conditional distributions, uniform on triangle
ef-200-86-00

Demonstration file: Conditional distributions, Bergengoc bulbs
ef-200-87-00

Demonstration file: Conditional distributions, standard normal
ef-200-88-00

Demonstration file: Conditional distributions, normal
ef-200-89-00

PROBLEMS

13. *Two dice, largest and smallest*
Toss 2 dice and let X be the largest and Y be the smallest of the two numbers. Find out (make a numerical table for)

    (a) the distribution of (X, Y);

    (b) the conditional distributions of Y on condition that X is given, and

    (c) the conditional distributions of X on condition that Y is given.

14. *Working with conditional distributions, discrete case*
Assume that X follows discrete uniform distribution between 1 and 10 (1 and 10 included), and if $X = x$, then Y follows discrete uniform distribution between 1 and x (1 and x included). Find the conditional distribution of Y on condition that

    (a) $X = 4$;
    (b) $X = 5$;
    (c) $X = x$.
    (d) What are the possible values of (X, Y) ? Find the distribution of (X, Y).

15. *Working with conditional distributions, discrete case*
Assume that X follows discrete uniform distribution between 1 and 10 (1 and 10 included), and if $X = x$, then Y follows discrete uniform distribution between 1 and x (1 and x included). What are the possible values of Y? Find the distribution of Y. Find the conditional distributions of X on condition that

    (a) $Y = 6$;
    (b) $Y = 7$;
    (c) $Y = y$.

16. *A particle landing in semi-circle*
Assume that a particle lands on the semi-circle $x^2 + y^2 \leq 1$, $y \geq 0$ according to uniform distribution, but we are able to observe only the X coordinate of the landing point (X, Y).

    (a) What is the conditional density function of the Y coordinate, if $X = 1/3$?
    (b) What is the conditional density function of the Y coordinate, if $X = x$? (Pay attention to give the domain of the conditional density function correctly.)

17. *Conditional density and distribution functions*
The density function of (X, Y) is $f(x, y) = 15xy^2 \ \ (0 < y < x < 1)$. Calculate and visualize by graphs of the density functions and distribution functions

    (a) $f_1(x)$;
    (b) $f_{2|1}(y|x)$;
    (c) $f_2(y)$;
    (d) $f_{1|2}(x|y)$;
    (e) $F_1(x)$;
    (f) $F_{2|1}(y|x)$;
    (g) $F_2(y)$;
    (h) $F_{1|2}(x|y)$.

18. *Calculating the conditional density functions*
Assume that the density function of the two-dimensional random variable (X, Y) is $f(x, y) = 6(y - x)$ $(0 < x < y < 1)$. Find

   (a) the density function of X, and

   (b) the conditional density function of Y on condition that $X = x$. (Pay attention to the correct definition of the domain of the formulas.)

19. *Working with conditional distributions, continuous case*
Let $X = \text{RND}_1$ and $Y = X * \text{RND}_2$.

   (a) Find the formula of the density function of X. (Denote the variable of the density function of X by x. Pay attention to set up the domain of the formula correctly.) Calculate the probabilities:

      i. $P(X < 0.1)$;
      ii. $P(X < 0.2)$;
      iii. $P(0.1 < X < 0.2)$.

   (b) Find the formula of the (conditional) density function of Y on condition that $X = 0.25$. Calculate the conditional probabilities:

      i. $P(Y < 0.1 \mid X = 0.25)$,
      ii. $P(Y < 0.2 \mid X = 0.25)$;
      iii. $P(0.1 < Y < 0.2 \mid X = 0.25)$.

   (c) Find the formula of the (conditional) density function of Y on condition that $X = 0.5$. Calculate the conditional probabilities:

      i. $P(Y < 0.1 \mid X = 0.5)$;
      ii. $P(Y < 0.2 \mid X = 0.5)$;
      iii. $P(0.1 < Y < 0.2 \mid X = 0.5)$.

   (d) Find the formula of the (conditional) density function of Y on condition that $X = 0.75$. Calculate the conditional probabilities:

      i. $P(Y < 0.1 \mid X = 0.75)$;
      ii. $P(Y < 0.2 \mid X = 0.75)$,
      iii. $P(0.1 < Y < 0.2 \mid X = 0.75)$.

   (e) Find the formula of the (conditional) density function of Y on condition that $X = x$. Calculate the conditional probabilities:

      i. $P(Y < 0.1 \mid X = x)$;
      ii. $P(Y < 0.2 \mid X = x)$,
      iii. $P(0.1 < Y < 0.2 \mid X = x)$.

   (In each case, denote the variable of the (conditional) density function of Y by y. Pay attention to set up the domain of the formula correctly.)

(f) Find the formula of the density function of (X, Y). (Denote the variables of the density function of (X, Y) by x and y. Pay attention to set up the domain of the formula correctly. Make a figure on the plane about the domain.) Calculate the (at least four of) following probabilities and conditional probabilities:

   i. $P(X + Y < 1)$;
   ii. $P(X + Y < 0.5)$;
   iii. $P(Y < 0.25)$;
   iv. $P(Y < 0.5)$;
   v. $P(Y < 0.75)$;
   vi. $P(Y < 0.5 * X)$;
   vii. $P(X * Y < 0.5)$;
   viii. $P(Y < 0.25 * X \bigcap X * Y < 0.5)$;
   ix. $P(Y < 0.25 * X \mid X * Y < 0.5)$;
   x. $P(X * Y < 0.5 \mid Y < 0.25 * X)$.

   (In each case, make a figure on the plane about the event(s) involved in that problem, and then set up the associated double integral(s), and then calculate the double integral(s).)

(g) You may make computer simulations, and then probabilities can be approximated by relative frequencies. Try to do so.

(h) Find the formula of the (unconditional) density function of Y so that you determine the distribution function of Y by calculating the probability $P(Y < y)$, and then you differentiate with respect to y.

20. *Determining density function* $f(x, y)$
Let us define a two-dimensional random variable as follows.

   (a) Let the horizontal coordinate X be the square-root of a random number, and let the vertical coordinate Y be uniformly distributed between 0 and X.

   (b) Let the horizontal coordinate X be the square-root of a random number, and let the vertical coordinate Y be uniformly distributed between 0 and X^2.

   (c) Let the horizontal coordinate X be the square-root of a random number, and let the vertical coordinate Y be X multiplied by the square-root of another random number.

   (d) Let the horizontal coordinate X be the cube-root of a random number, and let the vertical coordinate Y be uniformly distributed between 0 and X.

   (e) Let the horizontal coordinate X follow exponential distribution with parameter 1, and let the vertical coordinate Y follow exponential distribution with parameter X.

(f) Let the horizontal coordinate X follow exponential distribution with parameter 1, and let the vertical coordinate Y follow exponential distribution with parameter $1/X$.

(g) Let the horizontal coordinate X follow standard normal distribution, and let the vertical coordinate Y follow normal distribution with parameters $X/2$ and 1.

(h) Let the horizontal coordinate X follow standard normal distribution, and let the vertical coordinate Y follow normal distribution with parameters $2X$ and 1.

Make 1000 experiments and visualize the point-cloud. What is the two-dimensional density function $f(x,y)$?

5 Normal distributions in two-dimensions

EXCEL

The following files show two-dimensional normal random variables:

Demonstration file: Height and weight
ef-200-65-00

Demonstration file: Height and weight, ellipse, eigen-vectors (projections and conditional distributions are also studied)
ef-200-82-00

Demonstration file: Two-dim normal distributions normal distributions, projections and conditional distributions
ef-200-83-00

Demonstration file: Measuring voltage
ef-200-66-00

PROBLEMS

21. *Expected values, standard deviations, probabilities*
Let us consider the two-dimensional random variable (X,Y), which follows normal distribution with parameters $\mu_1 = 26$, $\sigma_1 = 4$, $\mu_2 = 14$, $\sigma_2 = 2$, $r = 0.6$.

(a) How much is the expected value and the standard deviation of X?

(b) How much is the probability that $22 < X < 26$?

(c) How much is the probability that $24 < X < 28$?

(d) How much is the expected value and the standard deviation of Y?

(e) How much is the probability that $12 < Y < 14$?

(f) How much is the probability that $13 < Y < 15$?

22. *Conditional expected values, standard deviations, probabilities*
Let us consider the two-dimensional random variable (X, Y), which follows normal distribution with parameters $\mu_1 = 26$, $\sigma_1 = 4$, $\mu_2 = 14$, $\sigma_2 = 2$, $r = 0.6$. How much is the conditional expected value and the conditional standard deviation of Y on condition that

    (a) $X = 25$?

    (b) $X = 26$?

    (c) $X = 27$?

    (d) $X = x$?

23. *Conditional expected values, standard deviations, probabilities*
Let us consider the two-dimensional random variable (X, Y), which follows normal distribution with parameters $\mu_1 = 26$, $\sigma_1 = 4$, $\mu_2 = 14$, $\sigma_2 = 2$, $r = 0.6$. How much is the probability that $12 < Y < 14$ on condition that

    (a) $X = 25$?

    (b) $X = 26$?

    (c) $X = 27$?

24. *Conditional expected values, standard deviations, probabilities*
Let us consider the two-dimensional random variable (X, Y), which follows normal distribution with parameters $\mu_1 = 26$, $\sigma_1 = 4$, $\mu_2 = 14$, $\sigma_2 = 2$, $r = 0.6$. How much is the probability that $13 < Y < 15$ on condition that

    (a) $X = 25$?

    (b) $X = 26$?

    (c) $X = 27$?

6 Independence of random variables

EXCEL

The following file serves to understand the notion of dependence and independence:

Demonstration file: Lengths dependent or independent
ef-200-91-00

PROBLEMS

25. *Independent binomially distributed random variables*
Assume that X and Y are independent of each other, and X follows binomial distribution with parameters 7 and 0.5, and Y follows binomial distribution with parameters 5 and 0.3. Determine the distribution of (X,Y).

26. *Passenger car and bus*
Assume that when a 5 passenger car has an accident, then the number X of injured people, independently of any other factors, has the following distribution:

$P(X = 0) = 0.4;$
$P(X = 1) = 0.2;$
$P(X = 2) = 0.1;$
$P(X = 3) = 0.1;$
$P(X = 4) = 0.1;$
$P(X = 5) = 0.1$,

and when an 8 passenger bus has an accident, then the number Y of injured people, independently of any other factors, has the following distribution:

$P(Y = 0) = 0.50;$
$P(Y = 1) = 0.10;$
$P(Y = 2) = 0.10;$
$P(Y = 3) = 0.05;$
$P(Y = 4) = 0.05;$
$P(Y = 5) = 0.05;$
$P(Y = 6) = 0.05;$
$P(Y = 7) = 0.05;$
$P(Y = 8) = 0.05.$
When a 5 passenger car hits an 8 passenger bus, then

(a) what is the probability that nobody is injured?
(b) what is the probability that exactly 1 person is injured?
(c) what is the probability that exactly 2 persons are injured?
(d) how much is the expected value of the the total number of injured people?
(e) what is the probability that exactly 2 persons are injured in the car on condition that the total number of injured people is 5?

27. *Working with two-dimensional density functions*
For each of the cases below, the density function $f(x, y)$ of a two-dimensional random point (X, Y) is defined. In each case, determine the density function $f_1(x)$ of X and the density function $f_2(x)$ of Y, and then decide whether X and Y are independent or not.

(a) $f(x, y) = 1 \quad (0 < x < 1, \;\; 0 < y < 1)$;
(b) $f(x, y) = 1/6 \quad (0 < x < 3, \;\; 0 < y < 2)$;
(c) $f(x, y) = 4xy \quad (0 < x < 1, \;\; 0 < y < 1)$;
(d) $f(x, y) = 6xy^2 \quad (0 < x < 1, \;\; 0 < y < 1$;
(e) $f(x, y) = e^{-x-y} \quad (x > 0, \;\; y > 0)$;
(f) $f(x, y) = 10e^{-5x-2y} \quad (x > 0, \;\; y > 0$;
(g) $f(x, y) = 2 \quad (x > 0, \;\; y > 0, \;\; 0 < x + y < 1)$;
(h) $f(x, y) = 2 \quad (0 < x < y < 1)$;
(i) $f(x, y) = 6(y - x) \quad (0 < x < y < 1)$;
(j) $f(x, y) = 3x \quad (0 < x < y < 1)$;
(k) $f(x, y) = 3(1 - y) \quad (0 < x < y < 1)$.

7 Generating a two-dimensional random variable

PROBLEMS

28. *Generating first X, then Y*
Consider the distribution on the plane with the density function

$$f(x, y) = \frac{2y}{x^2}\text{ , if } 0 < y < x < 1.$$

Simulate a two-dimensional random variable (X, Y), which follows this distribution, as follows:

(a) Determine $f_1(x)$, $F_1(x)$, $F_1^{-1}(u)$,
(b) then simulate X as $F_1^{-1}(\text{RND}_1)$.
(c) Determine $f_{2|1}(y|x)$, $F_{2|1}(y|x)$, $F_{2|1}^{-1}(v|x)$,
(d) then simulate Y as $F_{2|1}^{-1}(\text{RND}_2|X)$.
(e) Making 1000 experiments for (X, Y), construct a point-cloud of the 1000 experimental results, and check that the density suggested by the point-cloud in the triangle defined by the inequalities $0 < y < x < 1$ is in accordance with the density $f(x, y)$.

29. *Generating first Y, then X*
Consider the distribution on the plane with the density function

$$f(x,y) = \frac{2y}{x^2} \text{ , if } 0 < y < x < 1.$$

Simulate a two-dimensional random variable (X,Y), which follows this distribution, as follows:

(a) Determine $f_2(y)$, $F_2(y)$, $F_2^{-1}(v)$;
(b) then simulate Y as $F_2^{-1}(\text{RND}_2)$.
(c) Determine $f_{1|2}(x|y)$, $F_{1|2}(x|y)$, $F_{1|2}^{-1}(u|y)$;
(d) then simulate X as $F_{1|2}^{-1}(\text{RND}_1|X)$.
(e) Making 1000 experiments for (X,Y), construct a point-cloud of the 1000 experimental results, and check that the density in the region defined by the inequalities $0 < x < y$ is in accordance with the density $f(x,y)$.

30. *Generating first X, then Y*
Consider the distribution on the plane with the density function

$$f(x,y) = \frac{4y}{x} \text{ , if } 0 < y < x < 1.$$

Simulate a two-dimensional random variable (X,Y), which follows this distribution, as follows:

(a) Determine $f_1(x)$, $F_1(x)$, $F_1^{-1}(u)$;
(b) then simulate X as $F_1^{-1}(\text{RND}_1)$.
(c) Determine $f_{2|1}(y|x)$, $F_{2|1}(y|x)$, $F_{2|1}^{-1}(v|x)$;
(d) then simulate Y as $F_{2|1}^{-1}(\text{RND}_2|X)$.
(e) Making 1000 experiments for (X,Y), construct a point-cloud of the 1000 experimental results, and check that the density suggested by the point-cloud in the triangle defined by the inequalities $0 < y < x < 1$ is in accordance with the density $f(x,y)$.

31. *Generating first X, then Y*
Consider the distribution on the plane with the density function

$$f(x,y) = e^{-y} \text{ , if } 0 < x < y.$$

Simulate a two-dimensional random variable (X,Y), which follows this distribution, as follows:

(a) Determine $f_1(x)$, $F_1(x)$, $F_1^{-1}(u)$;
(b) then simulate X as $F_1^{-1}(\text{RND}_1)$.
(c) Determine $f_{2|1}(y|x)$, $F_{2|1}(y|x)$, $F_{2|1}^{-1}(v|x)$;

(d) then simulate Y as $F_{2|1}^{-1}(\text{RND}_2|X)$.

(e) Making 1000 experiments for (X, Y), construct a point-cloud of the 1000 experimental results, and check that the density in the region defined by the inequalities $0 < x < y$ is in accordance with the density $f(x, y)$.

32. *Water level in a water reservoir*
Let us assume that the water level of a water reservoir is measured in such a scale that the minimum level corresponds to 0, the maximum level corresponds to 1. Let us denote the water level on a day by X, and the water level two days later by Y. Let us assume that the two-dimensional random variable (X, Y) has the following density function:

$$f(x, y) = \frac{4}{5}(1 + xy) \quad \text{if} \quad 0 < x < 1, \;\; 0 < y < 1$$

Simulate (X, Y).

Solution: The density function of X is:

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^1 \frac{4}{5}(1 + xy)\, dy =$$

$$= \frac{4}{5}\left(1 + \frac{x}{2}\right) = \frac{4}{5} + \frac{2}{5}x \quad \text{if} \quad 0 < x < 1$$

The distribution function of X is:

$$F_1(x) = \int_{-\infty}^{x} f_1(x)\, dx = \int_0^x \frac{4}{5}\left(1 + \frac{x}{2}\right)\, dx$$

$$= \frac{4}{5}\left(x + \frac{x^2}{4}\right) = \frac{4}{5}x + \frac{1}{5}x^2 \quad \text{if} \quad 0 < x < 1$$

The inverse of the distribution function is the solution for x of the equation $F_1(x) = u$, which is now the following quadratic equation:

$$\frac{4}{5}x + \frac{1}{5}x^2 = u$$

The solution is

$$x = -2 + \sqrt{4 + 5u}$$

thus X can be simulated as

$$X = -2 + \sqrt{4 + 5\text{RND}_1}$$

The conditional density function of Y on condition that $X = x$ is:

$$f_{2|1}(y|x) = \frac{f(x, y)}{f_1(x)} = \frac{\frac{4}{5}(1 + xy)}{\frac{4}{5}\left(1 + \frac{x}{2}\right)} = \frac{1 + xy}{1 + \frac{x}{2}}$$

The conditional distribution function of Y on condition that $X = x$ is:

$$F_{2|1}(y|x) = \int_{-\infty}^{y} f_{2|1}(y|x)\, dy = \int_{0}^{y} \frac{1 + xy}{1 + \frac{x}{2}}\, dy = \frac{y + x\frac{y^2}{2}}{1 + \frac{x}{2}}$$

$$= \left(\frac{2}{2 + x}\right) y + \left(\frac{x}{2 + x}\right) y^2 \quad \text{if} \quad 0 < x < 1, \quad 0 < y < 1$$

The inverse of the distribution function of the conditional distribution is the solution for y of the equation $F_{2|1}(y|x) = v$, which is now the following quadratic equation:

$$\left(\frac{2}{2 + x}\right) y + \left(\frac{x}{2 + x}\right) y^2 = v$$

or equivalently

$$xy^2 + 2y - (2 + x)v = 0$$

The solution is

$$y = \frac{-2 + \sqrt{4 + 4x(2 + x)v}}{2x}$$

thus Y can be simulated as

$$Y = \frac{-2 + \sqrt{4 + 4X(2 + X)\mathrm{RND}_2}}{2X}$$

Solution
Sol-04-07-01

8 Properties of the expected value, variance and standard deviation

EXCEL

The following file shows the "average"-property of the standard deviation.

Demonstration file: Standard deviation of the average
ef-200-61-00

PROBLEMS

33. *Square root law for the standard deviation of the sum*
Let us assume that X and Y are independent random variables, and X follows normal distribution with parameters $\mu_1 = 10$ and $\sigma_1 = 3$, and Y follows normal distribution with parameters $\mu_2 = 15$ and $\sigma_2 = 4$. Calculate the probabilities:

    (a) $P(X + Y < 20)$;
    (b) $P(X + 2Y < 50)$;
    (c) $P(X - Y < -5)$;
    (d) $P(2X - Y < 5)$;

34. *Married couples*
Assume that the weight of a randomly chosen Hungarian woman is normally distributed with an expected value of 60 kg and standard deviation 4 kg, and the weight of a randomly chosen Hungarian man is normally distributed with an expected value of 75 kg and standard deviation 7 kg. Let us also assume that the weights of the wife and of the husband are independent of each other.

    (a) How much is the expected value and the standard deviation of
        i. the total weight of a randomly chosen married couple?
        ii. the total weight of 10 randomly chosen married couples?
        iii. the average weight of 10 randomly chosen men?
        iv. the average weight of 100 randomly chosen men?
        v. the average weight of 10 randomly chosen women?
        vi. the average weight of 100 randomly chosen women?
    (b) What is the probability that
        i. the the total weight of a randomly chosen married couple is between 130 and 140 kg?
        ii. the total weight of 10 randomly chosen married couples is between 1300 and 1400 kg?
        iii. the average weight of 10 randomly chosen men is between 74 and 76 kg?
        iv. the average weight of 100 randomly chosen men is between 74 and 76 kg?
        v. the average weight of 10 randomly chosen women is between 59 and 61 kg?
        vi. the average weight of 100 randomly chosen women is between 59 and 61 kg?
    (c) How much must n be in order that
        i. the average weight of n randomly chosen men is between 74 and 76 kg with a probability 0.9?
        ii. the average weight of n randomly chosen men is between 74 and 76 kg with a probability 0.99?

iii. the average weight of n randomly chosen women is between 59 and 61 kg with a probability 0.9?
iv. the average weight of n randomly chosen women is between 59 and 61 kg with a probability 0.99?

35. *Expected value, variance and standard deviation of sums of random numbers*
Determine the expected value, the variance and the standard deviation of the following random variables:

(a) RND;

(b) $RND_1 + RND_2$;

(c) $RND_1 + RND_2 + RND_3 + RND_4 + RND_5 + RND_6 + RND_7 + RND_8 + RND_9 + RND_{10} + RND_{11} + RND_{12}$.

36. *Continuation of the previous problem*
How many measurements are needed to guarantee that the average of the experimental results is in the interval $[6.4; 6.6]$ with a probability

(a) 0.99;

(b) 0.999 ?

37. *Non-independent random variables*
Try to define (by simulation) non-independent random variables so that the variance of the sum is not necessarily equal to the sum of the variance of the terms.

38. *Measuring voltages*
Let us assume that the voltage X (measured in Volt-s) in a certain electrical circuit follows normal distribution with parameters $\mu = 220$ and $\sigma = 2.5$. Using Excel, sketch the graph of the density function and of the distribution function of X.

(a) Using Excel, make 1000 experiments for X, and visualize them on a nice figure.

(b) $P(119 < X < 221) =?$

(c) $P(118 < X < 222) =?$

Assume that 5 independent measurements are performed for the voltage. Let $\overline{X}_5$ mean their average. Accept the fact that $\overline{X}_5$ follows a normal distribution, too.

(a) Using Excel, sketch the graph of the density function and of the distribution function of $\overline{X}_5$.

(b) Using Excel, make 1000 experiments for $\overline{X}_5$, and visualize them on a nice figure.

(c) $P(119 < \overline{X}_5 < 221) =?$

(d) $P(118 < \overline{X}_5 < 222) =?$

Assume that 50 independent measurements are performed for the voltage. Let $\overline{X}_{50}$ mean their average.

(a) How much is the expected value of $\overline{X}_{50}$?

(b) How much is the standard variation of $\overline{X}_{50}$?

Accept the fact that $\overline{X}_{50}$ follows a normal distribution, too.

(a) Using Excel, sketch the graph of the density function and of the distribution function of $\overline{X}_{50}$.

(b) Using Excel, make 1000 experiments for $\overline{X}_{50}$, and visualize them on a nice figure.

(c) $P(119 < \overline{X}_{50} < 221) =?$

(d) $P(118 < \overline{X}_{50} < 222) =?$

39. *CO_2 pollution*
The CO_2 pollution is measured on the main square of a town through 25 days. The 25 measurement results are considered independent of each other. At each measurement the probability that the CO_2 pollution is above the acceptance level is 1/5.

(a) Using normal distribution find the approximate value of the probability that during 25 days at least 6 times the CO_2 pollution is above the acceptance level.

(b) Assume that the amount of CO_2 pollution has a uniform distribution on the interval $[0.2]$. Using normal distribution find the approximate value of the probability that during 25 days the average of the measurement results is greater than az $1 + \frac{\sqrt{3}}{10} \approx 1.17$.

9 Transformation from plane to line

EXCEL

The following files show transformations from plane to line.

Demonstration file: Transformation from square to line by product
ef-300-02-00

Demonstration file: Transformation from square to line by ratio
ef-300-03-00

Demonstration file: Transformation from plane into chi distribution
ef-300-04-00

Demonstration file: Transformation from plane into chi-square distribution
ef-300-05-00

The following files show projections from plane to axes.

Demonstration file: Projection from triangle onto axes: $(max(RND_1; RND_2); min(RND_1; RND_2))$
ef-300-06-00

Demonstration file: Projection from triangle onto axes: $(RND_1; RND_1 RND_2)$
ef-300-07-00

Demonstration file: Projection from sail onto axes: $(RND_1 RND_2; RND_1/RND_2)$
ef-300-08-00

Demonstration file: Projection from sale onto axes: $(\sqrt{RND_1 RND_2}; \sqrt{RND_1/RND_2})$
ef-300-09-00

PROBLEMS

40. *Exponential random variables with different parameters*
X and Y are independent, exponential random variables with parameters λ_1 and λ_2 (life-times of electrical bulbs of different types). Let $Z = X + Y$. Calculate and visualize the distribution function and the density function of Z.

41. *Exponential random variables with equal parameters*
X and Y are independent, exponential random variables with the same parameter λ (life-times of electrical bulbs of the same type). Let $Z = X + Y$. Calculate and visualize by graphs the distribution function and the density function of Z.

42. *Transformation from plane to line*
The density function of (X, Y) is $f(x, y) = 15xy^2 \quad (0 < y < x < 1)$. Let $Z = X + Y$. Calculate and visualize by graphs the distribution function and the density function of Z. Calculate the expected value, the variance and standard deviation of Z without using the density function of Z. Calculate the expected value, the variance and standard deviation of Z using the density function of Z.

43. *Transforming a uniform distribution from a rectangle to a line*
Let X be uniformly distributed between 0 and 2, Y uniformly distributed between 0 and 3, X and Y independent and $Z = X + Y$.

    (a) Calculate the probability that $Z > 2$.
    (b) Find the expected value of Z.
    (c) Find the standard deviation of Z.

44. *Transformation from plane to line*
The density function of (X, Y) is $f(x, y) = 15xy^2 \quad (0 < y < x < 1)$. Let $Z = X + Y$. Calculate and visualize by graphs the distribution function and the density function of Z. Calculate the expected value, the variance and standard deviation of Z without using the density function of Z. Calculate the expected value, the variance and standard deviation of Z using the density function of Z.

10 *** Transformation from plane to plane

EXCEL

The following files give some transformation from plane to plane.

Demonstration file: Transformation from square onto a "sail"
ef-300-09-50

Demonstration file: Linear transformation of the standard normal point-cloud
ef-300-10-00

Demonstration file: Linear transformation of normal distributions
ef-300-11-00

PROBLEMS

45. *Transformation from square to triangle*
Consider the uniform distribution on the unit square,

$$\{(x, y) : 0 < x < 1,\ 0 < y < 1\}$$

which has the constant density function

$$f(x, y) = 1$$

Transform it by the transformation $u = xy$, $v = y$. Find the new density function.

46. *Transformation from square to triangle*
Consider the continuous distribution on the unit square

$$\{(x, y) : 0 < x < 1,\ 0 < y < 1\}$$

with the density function

$$f(x, y) = 4xy$$

Transform it by the transformation $u = xy$, $v = y$. Find the new density function.

11 *** Sums of random variables. Convolution

EXCEL

The following files show how the distribution of the sum can be calculated:

Demonstration file: Summation of independent random variables, fair dice ef-300-12-00

Demonstration file: Summation of independent random variables, unfair dice ef-300-13-00

PROBLEMS

47. *Getting a "triangle-shaped" distribution*
Consider the uniform distribution on the sets $0, \ldots, 5$. Convolve it by itself. You should get a "triangle-shaped" distribution on the set $0, \ldots, 10$. Making the numerical calculations, you may use Excel.

48. *Getting a "triangle-shaped" distribution*
Consider the uniform distribution on the sets $0, \ldots, n$. Convolve it by itself. You should get a "triangle-shaped" distribution on the set $0, \ldots, 2n$.

49. *Getting a "trapezoid-shaped" distribution*
Consider the uniform distributions on the sets $0, \ldots, n_1$ and $0, \ldots, n_2$, where $n_1 \neq n_2$. Convolve them. You should get a "trapezoid-shaped" distribution on the set $0, \ldots, n_1 + n_2$. Making the numerical calculations, you may use Excel.

50. *Getting a "trapezoid-shaped" distribution*
Consider the uniform distributions on the sets $0, \ldots, 5$ and $0, \ldots, 8$. Convolve them. You should get a "trapezoid-shaped" distribution on the set $0, \ldots, 13$.

51. *Convolving binomial distributions*
Convolve the binomial distribution with parameters 2 and 0.5 by the binomial distribution with parameters 3 and 0.5. You should get the binomial distribution with parameters 5 and 0.5. Making the numerical calculations, you may use Excel.

52. *Convolving binomial distributions*
Convolve the binomial distribution with parameters 2 and p by the binomial distribution with parameters 3 and p. You should get the binomial distribution with parameters 5 and p.

53. *Convolving Poisson-distributions*
Convolve the Poisson-distribution with parameter 2 by the Poisson-distribution with parameter 3. You should get the Poisson-distribution with parameter 3. Making the numerical calculations, you may use Excel.

54. *Convolving Poisson-distributions*
Convolve the Poisson-distribution with parameter λ_1 by the Poisson-distribution with parameter λ_2. You should get the Poisson-distribution with parameter $\lambda_1 + \lambda_2$.

12 Limit theorems to normal distributions

EXCEL

Here is a file to study the binomial approximation of normal distribution:

Demonstration file: Binomial approximation of normal distribution
ef-300-14-00

Here is a files to study how convolutions approximate normal distributions:

Demonstration file: Convolution with uniform distribution
ef-300-15-00

Demonstration file: Convolution with asymmetrical distribution
ef-300-16-00

Demonstration file: Convolution with U-shaped distribution
ef-300-17-00

Demonstration file: Convolution with randomly chosen distribution
ef-300-18-00

This is a file to study how gamma distributions approximate normal distributions:

Demonstration file: Gamma distribution approximates normal distribution
ef-300-19-00

Here are files to study the two-dimensional central limit theorem:

Demonstration file: Two-dimensional central-limit theorem, rectangle
ef-300-20-00

Demonstration file: Two-dimensional central-limit theorem, parallelogram
ef-300-21-00

Demonstration file: Two-dimensional central-limit theorem, curve
ef-300-22-00

PROBLEMS

55. Draw with replacement 25 times from the box containing the tickets with the numbers 3, 5, 7, 9 so that each ticket has the same chance.
    (a) What is the expected value of the sum of the 25 draws?
    (b) What is the standard deviation of the sum of the 25 draws?
    (c) What is the probability that the sum of the 25 draws is more than 140 ? (Use normal approximation.)
56. 400 independent random numbers which are uniformly distributed between 0 and 1 are generated. What is the (approximate) probability that their average is between 0.45 and 0.55? Give your answer using normal approximation.
57. Toss a die 600 times. What is the probability that the number of sixes is
    (a) greater than or equal to 120?
    (b) less than or equal to 80?
    (c) strictly between 80 and 120 ?
58. Assume that in a country 2/3 of the people are for party "A", and 1/3 of them are for party "B". Choosing 500 people at random, what is the probability that the relative frequency of the people being for party "A" among the chosen ones is between $\frac{2}{3} - 0.1$ and $\frac{2}{3} + 0.1$?
59. The diameter of a first class tomato sold in a certain shop is uniformly distributed between 6 and 9 cm, the diameter of a second class tomato is uniformly distributed between 4 and 7 cm. (The diameters of the tomatoes are independent.) If we put 25 first class tomatoes and 25 second class tomatoes in a line, then how much is the expected value and the standard deviation of the total length of the 50 tomatoes being in the line?
60. From the respondents who participated in an opinion survey, 50 percents declared to favor a unicameral parliament, 40 percents declared to favor a bicameral parliament, and 10 percents did not answer. 400 respondents from the interviewed sample are randomly chosen (For simplicity, you may assume: with replacement).

(a) What is the probability that exactly 200 of them were in favor of a unicameral parliament?

(b) What is the approximate probability that the number of respondents who favor a unicameral parliament is at least 190 and at most 210?