Markov processes and martingales
1st midterm test

22nd Mar 2021

1. Let S_n denote the simple symmetric random walk, that is, $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ is an iid. sequence of random variables distributed as $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

   (a) (2 points) Compute $\mathbf{E}(S_{n+1}^3 | \mathcal{F}_n)$ by using that $S_{n+1} = S_n + X_{n+1}$.

   (b) (2 points) Check that $M_n = S_n^3 - 3nS_n$ is a martingale adapted to the filtration $\mathcal{F}_n$.

   (c) (2 points) For two positive integers a, b, define the hitting times $T_{-a} = \inf\{n : S_n = -a\}$, $T_b = \inf\{n : S_n = b\}$ and $T = \min\{T_{-a}, T_b\}$. Apply the optional stopping theorem[1] for M_n and T to conclude that $\mathbf{E}(M_T) = 0$.

   (d) (2 points) By computing $\mathbf{E}(S_T)^2$ directly, determine the covariance of T and S_T. The facts $\mathbf{P}(T_{-a} < T_b) = b/(a+b)$ and $\mathbf{E}(T) = ab$ can be used without a proof.

2. In a population of fishes let X_1 denote the weight of a fish in grams and let X_2 be its speed in km/h. The random vector $\mathbf{X} = (X_1, X_2)$ has bivariate normal distribution where the weight has mean 80 and variance 100, the speed has mean 5 and variance 4 and the two components have correlation $-1/2$.

   (a) (1 point) Write down the covariance matrix of the vector $\mathbf{X}$.

   (b) (3 points) What is the distribution of the speed of a fish in the population which has weight 70 g? Recall from the lecture that the parameters of the conditional normal distribution are $\mu_{X|Y} = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(Y - \mu_Y)$ and $\Sigma_{X|Y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$.

   (c) (3 points) For which value of the parameter $p \in \mathbb{R}$ is the variance of $Y_p = 2X_1 + pX_2$ minimal. For that value of p, what is the distribution of Y_p?

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[1] The conditions of the optional stopping theorem are not satisfied because $M_{n \wedge T}$ does not have bounded increments, dominate convergence can be used instead.

[2] It should have been $\mathbf{E}(S_T^3)$.