TDK PAPER

Random Walk in Periodic Environment

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1 Introduction

In the last three decades Random Walk in Random Environment (RWRE) has been a core subject of intensive research in probability theory. Historically among the first influential papers Sinai's article has to be mentioned, see [2]. One of the main reasons is perhaps that these phenomena possess in general more complex problems, and for their studies development of new technical tools is almost inevitable. A comprehensive introduction to this topic with an extensive bibliography is Zeitouni's lecture notes [6].

In general for the definition of RWRE two components are needed. First, in each steps the random walker is given a randomly chosen environment and second, the random walk is a Markov chain whose transition probabilities are defined by the environment chosen, by this way the transition matrices change step by step.

1.1 Mathematical definition

Let $d \geq 1$. Define a collection of probability measures $\mathcal{M}$ with finite support on $\mathbb{Z}^d$, that is in our case, the *nearest neighbor* RWRE $\{e \in \mathbb{Z}^d : |e| = 1\}$. An environment that describes the transition probabilities is an element $\omega \in \Omega := \mathcal{M}^{\mathbb{Z}^d}$, $\omega = (\omega_x)_{x \in \mathbb{Z}^d}$. $\mathcal{F}$ denote the corresponding σ-algebra on Ω. We define the random walk in random environment as a Markov chain $\{X_n\}$ with transition probabilities

$$P_\omega(X_{n+1} = x + y | X_n = x) := \omega_x(y).$$

Let us denote the law of a Markov chain with these transition probabilities and with the starting point x by $P_{x,\omega}$. We write Γ for the set of paths in $\mathbb{Z}^d$, with the appropriate σ-algebra $\mathcal{G}$. Clearly for any $G \in \mathcal{G}$, P defines a measurable mapping from $\mathbb{Z}^d \times \Omega$ to Γ, that is $(x, \omega) \mapsto P_{x,\omega}(G)$. We refer to $P_{x,\omega}$ as the *quenched law*.

To a randomly chosen sample environment ω we fix a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Naturally the measure $\mathbb{P}_x := P \otimes P_{x,\omega}$ on $(\Omega \times \Gamma)$ may also be defined,

$$\mathbb{P}_x(F \times G) = \int_F P_{x,\omega}(G)\mathbb{P}(d\omega).$$

It is common to call $\mathbb{P}_x$ the *annealed law*. Note that under $\mathbb{P}_x$ the RWRE $\{X_n\}$ is not a Markov chain. It is important to remark that throughout this paper only *quenched laws* will be studied.

1.2 Setup of our model

In this paper we study nearest neighbor random walk in periodic environment (RWPE), a special case of the general model described above. More precisely, fix the period $M := (M_1, \dots, M_d)$ and assume the environment is periodic, i.e. $\omega_x(y) = \omega_{x+(k_1M_1,k_2M_2,\dots,k_dM_d)}(y + (k_1M_1, k_2M_2, \dots, k_dM_d))$, where $k_1, \dots, k_d \in \mathbb{Z}^d$. We will also use the notation $p_x(e) = \omega_x(x + e)$ for $P(X_{n+1} = x + e|X_n = x)$, where $x \in \mathbb{Z}^d, |e| = 1$. So the period M can be thought of simply as a hypercube in $\mathbb{Z}^d$. In this setup the sample space is $T = \mathbb{Z}^d/M = [0, M_1) \times \cdots \times [0, M_d)$, where T is a d dimensional *torus*. Apart from the periodicity condition, for the random walk we assume that $p_x(e) \neq 0$ for $\forall x \in T$ and $\forall e$. So in this way the asymptotic velocity will be some positive constant, that is the random walk will be *ballistic*. In the sequel we will only deal two slightly different types of Markov chains.

Definition 1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain* ***defined on*** *$\mathbb{Z}^d$, with transition probabilities that have dependence on finite number of site determined by the fixed period M, i.e. with the notation above $\{p_x(e)\}$, where now $x \in \mathbb{Z}^d$.*

Let us denote this Markov chain by $\{X_n\}_{n\in\mathbb{N}}$

Definition 2. *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a Markov chain,* ***defined on*** *$T = \mathbb{Z}^d/M$, with transition probabilities that have dependence on finite number of site, i.e. $\{p_y(e)\}$, where $y \in T$ and $M \in \mathbb{Z}^d$ is fixed and denotes the period.*

Let us denote this Markov chain by $\{Y_n\}_{n\in\mathbb{N}}$

The crucial difference between $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ is that the later one is defined only on the d dimensional *torus*, whereas the sample space of $\{X_n\}_{n\in\mathbb{N}}$ is the **whole** $\mathbb{Z}^d$. These two Markov chains strongly rely on each other, in fact with a carefully defined function one can be mapped onto another and vice versa. Therefore introduce the equivalence relation $x \backsim y$, i.e. $x = y + k_1M_1 + k_2M_2 + \cdots + k_dM_d$. With these equivalence classes $\{Y_n\}_{n\in\mathbb{N}}$ can be uniquely determined if $\{X_n\}_{n\in\mathbb{N}}$ is given. The other direction is a bit harder, however still can be. A possible way could be the following. Assign the row vectors $\underline{k}^d(n) := (k_1, k_2, \dots, k_d)$ to each step of $\{Y_n\}_{n\in\mathbb{N}}$. A row vector $\underline{k}^d(n)$ tells exactly how many times the random walk has gone round on the *marginal* periods $k_{(\cdot)}$, by this way the position on $\mathbb{Z}^d$ is given by $X_n = Y_n + (k_1M_1, \dots k_dM_d)$. Further, since the assumption of $p_x(e) \neq 0$ for $\forall x \in T$ and $\forall e$ the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ and also $\{Y_n\}_{n\in\mathbb{N}}$ will be irreducible, therefore the *stationary measure* π of $\{Y_n\}_{n\in\mathbb{N}}$ will uniquely exists. In case of $\{X_n\}_{n\in\mathbb{N}}$ this is not true, i.e. a stationary measure may be given, but note that it will not be a probability measure, since its sample space $\mathbb{Z}^d$ is not finite.

For the purpose of study we require the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ to be reversible, i.e. $\{X_n\}_{n=0}^N \stackrel{d}{=} \{X_{N-n}\}_{n=0}^N$. Note that in case of $\{Y_n\}_{n\in\mathbb{N}}$ the reversibility condition will not hold. Define the potential as the mapping $u : S \to \mathbb{R}$ for which $\log\left(\frac{p_x(e)}{p_{x+e}(-e)}\right) = u(x) - u(x+e)$; hereafter log has base e. By the reversibility condition the potential will exist uniquely apart from addition of a constant, which is important since we intend to determine the relationship between the asymptotic direction of the potential and the limiting direction of the motion. Our conjecture arose intuitively, that is the angle enclosed by the asymptotic direction of the motion and the the asymptotic direction of the potential cannot be greater than a right angle. In Section 4 we show that this conjecture is proved to be correct.

The paper is organized as follows. In next Section we discuss the the Law of Large Numbers and Central Limit Theorem for RWPE in *quenched law* context. In Section 3 we discuss the $d = 1$ case. Next, in Section 4 our main result is presented. Finally Section 5 concludes.

2 LLN and CLT

In many studies in probability theory some essential questions can be addressed, two of these are whether the Law of Large Numbers and the Central Limit Theorem hold. Moreover if the answers are positive to these questions, then their related conditions and assumptions have to be specified as well.

These two questions are naturally arose in our RWPE model. In both cases the answer is positive. A slightly complicated proof of CLT for RWPE can be found in [3], however here we attempt to give a more simple way to prove it. Though we are not aware of any proof of CLT, it also does not take too much effort to prove it. Similarly in both cases widely known, fundamental results will be used.

In this section we work with a slightly modified Markov chain. The main difference in the definition lies the possible values the Markov chain can takes.

Definition 3. *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a Markov chain, so that $Z_n := (Y_{n-1}, Y_n - Y_{n-1})$ defined on $S := \mathbb{Z}^d/M \times \{e : |e| = 1\}$.*

Let us denote this Markov chain by $\{Z_n\}_{n\in\mathbb{N}}$

Remark 1. *The transition probabilities of $\{Z_n\}_{n\in\mathbb{N}}$*

$$P(Z_{n+1} = (x', e') | Z_n = (x, e)) = \begin{cases} p_x(e) & \text{if } x' = x + e \mod M \\ 0 & \text{otherwise,} \end{cases}$$

where M is the fixed period.

In this definition the only difference to the Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ is that $\{Z_n\}_{n\in\mathbb{N}}$ is defined on the *edges* of the d dimensional torus $T = \mathbb{Z}^d$. This Markov chain will not be reversible as well.

Proposition 1. *The Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ is irreducible.*

Proof. It is clear, since for the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ the assumption $p_x(e) \neq 0 \quad \forall x \in \Omega$ holds; $\{X_n\}_{n\in\mathbb{N}}$ is irreducible. Due to the the mappings between $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ and the definition of $\{Z_n\}_{n\in\mathbb{N}}$, the proposition follows. □

Here we point out that since the Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ has a stationary measure, $\{Z_n\}_{n\in\mathbb{N}}$ will also have a stationary measure, which can be defined as the stationary measure of $\{Y_n\}_{n\in\mathbb{N}}$ weighted by the transition probabilities, i.e. $\pi_Z(x, e) = \pi(x)p_x(e)$.

Theorem 1. *(**Ergodic theorem for Markov chains**) Let ξ_n be an irreducible Markov chain and π its stationary measure. If f is a real function on the state space S then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} f(\xi_i) = m \qquad a.s.$$

where

$$m = \mathbb{E}_\pi f = \sum_x \pi(x) f(x)$$

is the expectation of f with respect to π.

Proof. For the proof see [4]. □

LLN. Now we are ready to give the

Theorem 2. *Consider the stationary Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ with stationary measure π_Z determined by $\{X_n\}_{n\in\mathbb{N}}$. Let*

$$\nu := \mathbb{E}_{\pi_Z}(Z_1) = \sum_{z\in\Omega} \sum_{|e|=1} \pi(x) p_x(e) f((x, e)), \quad where \quad f((x, e)) = e.$$

Then

$$\frac{X_n}{n} \xrightarrow{a.s.} \nu, \quad P\left(\{\omega : \frac{X_n}{n} \to \nu\}\right) = 1$$

Proof. Since the irreducibility condition of *Theorem 1* holds and for the function $f : S \to \mathbb{R}$ given above

$$X_n = \sum_{k=1}^{n} f(Z_k),$$

since Z_n was defined on the edges of the d dimensional torus T, i.e. the edges are the differences of two consecutive steps of the Markov chain Y_n, the LLN is really a corollary of *Theorem 1.* □

The first steps for **CLT** we proceed as follows. Define the Hilbert-space $L^2(S, \pi)$ with the operator P on it by

$$(Pf)(x) = \sum_{y \in S} P_{xy} f(y),$$

which is the transition matrix. Suppose $m = 0$ i.e. $\mathbb{E}_\pi(f) = 0$. So, the variance

$$\begin{aligned}
\frac{1}{n}\mathbb{D}^2\left(\sum_{i=1}^{n} f(X_i)\right) &= \frac{1}{n}\mathbb{E}\left(\left(\sum_{i=1}^{n} f(X_i)\right)^2\right) \\
&= \frac{1}{n}\mathbb{E}\left(f(X_1)^2 + f(X_2)^2 + \dots f(X_n)^2\right) \\
&+ \frac{1}{n}2\mathbb{E}\left(f(X_1)f(X_2) + f(X_1)f(X_3) + \dots\right) \\
&\overset{\text{stac.}}{=} \frac{n}{n}\mathbb{E}\left(f(X_1)^2\right) \\
+2\left(\mathbb{E}\left(f(X_1)f(X_2)\right)\frac{n-1}{n} \right. &\left. + \mathbb{E}\left(f(X_1)f(X_2)\right)\frac{n-2}{n} + \dots\right) \\
\longrightarrow \mathbb{E}\left(f(X_1)^2\right) &+ 2\left(\mathbb{E}\left(f(X_1)f(X_2)\right) + \mathbb{E}\left(f(X_1)f(X_3)\right) + \dots\right) \\
= \mathbb{E}\left(f(X_1)^2\right) &+ 2\mathbb{E}\left(f(X_1)Pf(X_1)\right) + 2\mathbb{E}\left(f(X_1)P^2 f(X_1)\right) + \dots \\
&= \mathbb{E}\left(f(X_1)[I + 2P + 2P^2 + 2P^3 + \dots]f(X_1)\right) \qquad (1) \\
&= 2\mathbb{E}\left(f(X_1)[(I-P)^{-1}f](X_1)\right) - \mathbb{E}\left(f(X_1)^2\right). \qquad (2)
\end{aligned}$$

Since $m = 0$, f cannot be a constant. Together with the irreducibility implies that f is in the orthocomplement of the one dimensional eigenspace corresponding to the eigenvalue 1 of P. In this subspace $||P|| < 1$ and

$$I + 2P + 2P^2 + 2P^3 + \dots = 2(I-P)^{-1} - I, \Longrightarrow \ (2) = \ (1)$$

and $2(I-P)^{-1}-I$ is a bounded operator, since $m=0$, in this way we get that the expression in (2) is finite.

With this construction above we have

Theorem 3. *(**Markov chain Central Limit Theorem**) Let ξ_n be an irreducible Markov chain with its stationary measure π. If*

$$f: S \to \mathbb{R} \qquad \mathbb{E}_\pi(f) = \sum_{x\in S} \pi(x) f(x) = 0$$

Then

$$\frac{\sum_{i=1}^n f(\xi_i)}{\sqrt{n}} \Longrightarrow \mathcal{N}(0,\sigma^2).$$

Proof. Again, for the proof see [4]. □

Theorem 4. *For the Markov chain $\{X_n\}_{n\in\mathbb{N}}$*

$$\frac{X_n}{\sqrt{n}} \Longrightarrow \mathcal{N}(0,\sigma^2),$$

holds.

Proof. Argument goes similarly as above. Consider the Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ with its stationary measure $\pi_Z(x,e) = \pi(x)p_x(e)$. In order to apply *Theorem 3* we need to have

$$f: S \to \mathbb{R} \qquad \mathbb{E}_\pi(f) = \sum_{(x,e)\in S} \pi_Z(x,e) f(x,e) = 0.$$

To fulfil these conditions we only have to work with

$$f((x,e)) = e - \nu,$$

where $\nu = \mathbb{E}_{\pi_Z}(Z_1)$. By this way we get

$$\mathbb{E}_{\pi_Z}(f) = \sum_{(x,e)\in S} \pi(x) p_x(e) f(x,e) = 0$$

The irreducibility condition is satisfied as argued in the beginning of this section. The only thing remains to be seen is that

$$\sum_{i=1}^n f(Z_i) = X_n.$$

This is a clear consequence of $X_n - X_{n-1} = f(Z_n)$. Therefore CLT for RWPE is really a corollary of *Theorem 3*. □

3 One dimensional RWPE

In this section we study show analogous result, that is if the random walk has some drift in one direction, say in the right direction, then almost surely the particle will not be found on the other (the left-hand) side of the origin, when the number of steps, $n \to \infty$. Throughout this section we assume a fixed environment ω, in order to remain in '*quenched law*' context.

Let $X_0, X_1, X_2, \dots$ be a Markov chain defined on $\mathbb{Z}$ with site dependent transition probabilities

$$p_i = P(X_{n+1} = i+1 | X_n = i), \quad q_i = 1 - p_i = P(X_{n+1} = i-1 | X_n = i), \quad i \in \mathbb{Z}.$$

The random walk starts from the origin, i.e. $P(X_0 = 0) = 1$. That is we deal the with Markov chain $\{X_n\}_{n \in \mathbb{N}}$ in $d = 1$ defined in *Section 1.2*. So we also have a fixed period M.

Note that $\nu := \mathbb{E}(X_n) = \lim_{n \to \infty} \frac{X_n}{n}$ exists because of the LLN deduced in the previous section.

Theorem 5. *Consider the Markov chain* $\{X_n\}_{n \in \mathbb{N}}$ *in* $d = 1$.

(i) If $C(M) < 0 \Rightarrow \nu \geq 0 \qquad$ *a.s.;*

(ii) If $C(M) > 0 \Rightarrow \nu \leq 0 \qquad$ *a.s.,*

where $C(M) := \frac{1}{M} \sum_{j=1}^{M} \log \frac{q_{j+1}}{p_j}$ *is a constant depending on the parameters of the period* M.

Remark 2. Theorem 5 *is a special* $d = 1$ *case of our main result. We intend to generalize this theorem in higher dimension.*

We also introduce the notations we need in the sequel

$$\tau_{m_-} := \min\{k : X_k = m_-\}.$$

with $m_- < 0 < m_+$ Our main goal in this section is to prove the following theorem

Proposition 2. *Consider the Markov chain* $\{X_n\}_{n \in \mathbb{N}}$ *and* $d = 1$. *Let* $C(M)$ *be defined as above so that* $C(M) < 0$. *Then*

$$P(\min_{k \in \mathbb{N}} X_k < -n | X_0 = 0) = e^{C(M)n} \longrightarrow 0, \qquad as \quad n \to \infty.$$

Remark 3. *This proposition is the* (*i*) *part of* Theorem 5, *and here we will prove only this. The proof of part* (*ii*) *follows by symmetry.*

Here we point out that this problem is basically a generalized *gambler's ruin* type problem.

Proof of Proposition 2. Now fix an interval $[m_-, m_+]$ and $m_- < a < m_+$ We are interested in finding the probability that the particle reaches m_- first, the lower limit of the interval before hitting m_+, the upper limit of the interval

$$Q_{m_-,m_+}(a) := P\left(\tau_{m_-} < \tau_{m_+} | X_0 = a\right).$$

In order to compute $Q_{m_-,m_+}(a)$ we define the random variable

$$f(k) := 1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \frac{q_i}{p_i}, \qquad Z_n := f(X_n).$$

Proposition 3. *$\{Z_n\}$ is a martingale under the filtration $\{\mathcal{F}_{n-1}\}$.*

Proof. We need to see if $\mathbb{E}(Z_n|\mathcal{F}_{n-1}) = Z_{n-1}$ holds. Only a standard computation is required to this.

$$\begin{aligned}
\mathbb{E}(Z_n|\mathcal{F}_{n-1}) &= p_{X_{n-1}}(f(X_{n-1}+1)) + q_{X_{n-1}}(f(X_{n-1}-1)) \\
&= p_{X_{n-1}}\left(1 + \sum_{j=1}^{X_{n-1}} \prod_{i=1}^{j} \frac{q_i}{p_i}\right) + q_{X_{n-1}}\left(1 + \sum_{j=1}^{X_{n-1}-2} \prod_{i=1}^{j} \frac{q_i}{p_i}\right) \\
&= \left(1 + \sum_{j=1}^{X_{n-1}-2} \prod_{i=1}^{j} \frac{q_i}{p_i}\right)(p_{X_{n-1}} + q_{X_{n-1}}) + p_{X_{n-1}}\left(\prod_{i=1}^{X_{n-1}-1} \frac{q_i}{p_i} + \prod_{i=1}^{X_{n-1}} \frac{q_i}{p_i}\right) \\
&= \left(1 + \sum_{j=1}^{X_{n-1}-2} \prod_{i=1}^{j} \frac{q_i}{p_i}\right) + \left(\prod_{i=1}^{X_{n-1}-1} \frac{q_i}{p_i}\right)\left(1 + \frac{q_{X_{n-1}}}{p_{X_{n-1}}}\right)p_{X_{n-1}} \\
&= \left(1 + \sum_{j=1}^{X_{n-1}-1} \prod_{i=1}^{j} \frac{q_i}{p_i}\right) = Z_{n-1}
\end{aligned}$$

□

Now, with this martingale we can apply the classical theorem from [5].

Theorem 6. ***(Doob's Optional-Stopping Theorem)*** *Let T be a stopping time. Let X be a martingale. If any of the conditions*

(i) T is bounded (for some $N \in \mathbb{N}, T_\omega \leq N, \forall \omega$);

(ii) X is bounded (for some $K \in \mathbb{R}^+, |X_n(\omega)| \leq K$ for every n and ω) and T is a.s. finite;

(iii) $\mathbb{E}(T) \leq K$, *and, for some* $K \in \mathbb{R}^+$

holds, then

$$\mathbb{E}(X_T) = \mathbb{E}(X_0).$$

Proof. For the proof of the theorem see [5]. □

Since for Z_n the conditions in *Theorem 6* hold, we have

$$\mathbb{E}(Z_{m_- \wedge m_+}) = \mathbb{E}(Z_0)$$

$$Q_{m_-,m_+}(a)\left(1 - \sum_{j=m_-}^{a-1}\prod_{j=i}^{a-1}\frac{p_i}{q_i}\right) + (1 - Q_{m_-,m_+}(a))\left(1 + \sum_{j=a}^{m_+-1}\prod_{i=a}^{j}\frac{q_i}{p_i}\right) = 1$$

$$Q_{m_-,m_+}(a) = \frac{\sum_{j=a}^{m_+-1}\prod_{i=a}^{j}\frac{q_i}{p_i}}{\sum_{j=a}^{m_+-1}\prod_{i=a}^{j}\frac{q_i}{p_i} + \sum_{j=m_-}^{a-1}\prod_{j=i}^{a-1}\frac{p_i}{q_i}}.$$

Now, push the upper limit of the interval in right direction while the lower limit is being kept fixed, i.e. take the limit $m_+ \to \infty$

$$Q_{m_-,\infty}(a) = \frac{\sum_{j=a}^{\infty}\prod_{i=a}^{j}\frac{q_i}{p_i}}{\sum_{j=a}^{\infty}\prod_{i=a}^{j}\frac{q_i}{p_i} + \sum_{j=m_-}^{a-1}\prod_{j=i}^{a-1}\frac{p_i}{q_i}}.$$

The probability we need to find is

$$P(\min_{k\in\mathbb{N}} \quad X_k < -n | X_0 = 0) = Q_{-n,\infty}(0)$$

We take the limit in $M \cdot n$ and then $n \to \infty$ this can be done for two reasons. First the periodicity of the environment. The second argument is that the tail distribution of τ_{-m} is exponentially small. Put it another way, if the limit of the interval $[m_-, m_+]$ is kept fixed, then

$$P(\{X_n\} \quad \text{never hits} \quad [m_-, m_+]^c) = 0$$

So, we have to show that $Q_{-Mn,\infty}(0)$ decays exponentially as $n \to \infty$. Take the logarithm

$$\lim_{n\to\infty}\frac{1}{Mn}\log Q_{-Mn,\infty}(0) = \lim_{n\to\infty}\frac{1}{Mn}\log\frac{\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{q_i}{p_i}}{\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{q_i}{p_i} + \sum_{j=-Mn}^{-1}\prod_{j=i}^{-1}\frac{p_i}{q_i}}$$

Due to the assumption the sum in the numerator and also the first term in the denominator is finite, we can write

$$\lim_{n\to\infty}\frac{1}{Mn}\log\left(\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{q_i}{p_i}\right) - \lim_{n\to\infty}\frac{1}{Mn}\log\left(\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{q_i}{p_i} + \sum_{j=-Mn}^{-1}\prod_{j=i}^{-1}\frac{p_i}{q_i}\right) \leq$$

and with $\lim_{n\to\infty} \frac{1}{Mn}$ the first term $\to 0$

$$\leq - \lim_{n\to\infty} \frac{1}{Mn} \log \left(\sum_{j=-Mn}^{-1} \prod_{j=i}^{-1} \frac{p_i}{q_i} \right) =$$

Since the single periods are independent,

$$- \lim_{n\to\infty} \frac{1}{Mn} \log \left(\prod_{j=1}^{M} \frac{p_j}{q_{j+1}} \right)^n = - \lim_{n\to\infty} \frac{1}{M} \sum_{j=1}^{M} \log \frac{p_j}{q_{j+1}} =$$

Hence we arrived to the desired result

$$\frac{1}{M} \sum_{j=1}^{M} \log \frac{q_{j+1}}{p_j} = C(M) < 0.$$

□

4 Two and higher dimensional RWPE

We arrived to present our main result in $d \geq 2$. The conjecture mentioned in the introduction part, will be proved to be correct. More precisely if reversibility is assumed for $\{X_n\}$, then the limiting velocity of the random walker and the asymptotic direction of the potential cannot enclose an angle $\alpha > \pi/2$. Before claim the result in mathematical detail, let us define the *gradient of the environment* by

$$\underline{g} := \left(\frac{1}{M_1} \log \prod_{i=0}^{M_1-1} \frac{p_{i+1}e_1(-e_1)}{p_{ie_1}(e_1)}, \frac{1}{M_2} \log \prod_{i=0}^{M_2-1} \frac{p_{i+1}e_2(-e_2)}{p_{ie_2}(e_2)} \dots \right)$$

which is basically the asymptotic direction of the potential. Recall $\nu = \lim_{n\to\infty} \frac{X_n}{n}$

Theorem 7. *Consider the reversible Markov chain $X_{n n\in\mathbb{Z}}$, and assume that $\underline{g} \neq \underline{0}$. Then $\langle \underline{g}, \underline{\nu} \rangle \geq 0$.*

Remark 4. *This theorem holds for any dimension. We have seen in $d = 1$ case, **now it will be proved in** $d = 2$. The proof goes similarly in higher dimension, with the necessary modification.*

Remark 5. *The idea is traced back to those discussed above in $d = 1$, we will try to grasp the direction of $\underline{g}$ through first hitting times. See* Theorem 8.

Now we gather the important tools and definitions. Take the line orthogonal to $\underline{g}$ across the origin $(0,0)$, that is $l_g := \{\underline{x} : \langle \underline{x}, \underline{g} \rangle = 0, \quad \underline{x} \in \mathbb{R}^2\}$, i.e. $g_o = \{c\underline{g}\, c \in \mathbb{R}$. Take the line $g_o := \{(x,y) : \langle (x,y), (x',y') \rangle = 0, \quad \text{where} \quad (x',y') \in l_g\}$. Translate the line l_g by $k \in \mathbb{Z}$ on g_o in both direction with an integer multiple of the period M, so we take $p \in g_o$ so that $p \sim (0,0)$ and consider

$$l_g^{\pm k} := \{\underline{x} : \langle \underline{x}, \underline{g} \rangle = \pm kc, \quad \underline{x} \in \mathbb{R}^2\},$$

where $c = \langle \underline{p}, \underline{g} \rangle$ is some fixed constant determined by the gradient of the environment. Important to note, that in the sequel $-k$ will mean the translation in the left direction, i.e. and k in the right direction. To put it another way, if one chooses an arbitrary element $(x_{A_{-k}}, y_{A_{-k}})$ from the half plane $A_{-k} := \{(x,y) : \langle (x,y), (g_x, g_y) \rangle < -kc \quad (x',y') \in g_o\}$, then $(x_{A_{-k}}, y_{A_{-k}})\underline{g} < 0$. Define the *level lines* in such a way that they are close to l_g^k, periodical and disconnect $\mathbb{Z}^2$, denote these level lines by $m_{\pm k}$. That is the level lines m_k are sets of points in $\mathbb{Z}^2$.

Analogously to $d = 1$ define the *first hitting times*

$$\tau_k := min\{n : X_n \in m_k\}.$$

Further assumption for now is that $\underline{g}_i \in \mathbb{Q}$ for $\forall i = 1, \ldots, d$. We deal with the remaining case only at a later stage.

With these definitions just given above, analogously to the $d = 1$ case, we claim

Theorem 8. *Consider the reversible Markov chain X_n in $d = 2$. Then*

$$P(X_n \in m_{-k}) = e^{C_2 k} \longrightarrow 0, \qquad as \quad k \to \infty,$$

where $C_2 < 0$ some constant depending on the parameters of the period M.

Proof. As already mentioned above, the idea of the proof share some link with the concept of the $d = 1$ case.

Since the environment is periodic and X_n is reversible the potentials and also the potential differences $\Delta_{a,b} := u(a) - u(b)$ along the central level line m_0 can only take a finite number of possible values and these will be periodically repeated. So to each level line m_k only finite number of possible potentials belong.

Our goal is to see that from an arbitrary chosen point $a \in m_0, a \in \mathbb{Z}^2$,

$$P^{-k} := P\left(\omega : \exists k \quad \tau_{-k} < \tau_k | X_0 = a \in m_0\right) <$$
$$P\left(\omega : \exists k \quad \tau_{-k} > \tau_k | X_0 = a \in m_0\right) =: P^{+k},$$

in other words, we want to show that the probability of hitting the level m_{-k} provided starting from an arbitrary chosen, but fixed point on the central level line is strictly smaller than reaching the level line m_k, with the same starting condition.

It is one of the key idea of the proof, since if we have that $P^{-k} < P^k$ then we have a well defined random walk on the level lines $\ldots, m_{-k_1}, m_0, m_{k_1}, \ldots,$ for which the theorem holds, as we have seen the results in *Section 3*. Though on these level lines the random walk will have random time instants, i.e. the time needed for jumping to the nearest neighbor level line is not deterministic. However, since the LLN this time will have finite expectation. Moreover the tail distribution of $\tau_{+k} \wedge \tau_{-k}$ decays exponentially fast. Also it follows from the periodicity that the time instants between two neighbor levels are independent.

To see that $P^{-k} < P^k$, we attempt to give estimates to the paths going from the central level line to the nearest neighbor level m_{-k}. It turns out that estimates can always be given by certain paths to the level line m_k constructed from the paths needed to be estimated.

In order to go on this way three different kind of points will be defined. Take a sample path ω from $a \in m_0$ to a fixed point $c = X_{\tau_{-k}}(\omega)$. In general it is possible that this path crosses some times the line m_0 before reach the level m_{-k}, however we are only interested in the last crossing point. Therefore define the last visiting point of the path on the central level line by

$$b := X_{\eta_b}(\omega), \qquad \text{where} \quad \eta_b = \max\{t : X_t(\omega) \in m_0 \quad \text{and} \quad 0 \le t < \tau_{-k}\}.$$

Hence the three kinds of points are: the starting point a, the last visiting point on the central level line b ($a = b$ is also allowed), and the hitting point c of level m_{-k}. This is a *canonical decomposition* can uniquely be made in case of each path from the central gradient to the next levels.

So the required estimation can be constructed in the following way. Fix an arbitrary starting point a, which is now assumed to be the origin $(0, 0)$. Take a sample path ω to the neighbor level m_{-k}, the end of this path will be the point $c = X_{\tau_{-k}}$. Along the sample path ω it is possible that there is a last visiting point b, which is now assumed to be $b \neq a$. Now the setup is given by a fixed sample path ω.

Let us *canonically* choose for $\forall \underline{x}, \underline{y} \in m_0 \cap T$ a path with some probability weight and denote this path by $\gamma_{x,y}$. The subscript tells us the direction of the path, i.e. it goes from $x \to y$. Important to note, that number of these weights are finite, since the periodicity of the environment.

Let us denote the path from a to b by

$$\omega_{a \to b} := \text{trajectory} \quad a \to b : X_0(\omega), X_1(\omega), \ldots, X_{\eta_b}(\omega),$$

and the reversed path

$$\overline{\omega}_{b\to a} := \text{trajectory} \quad b \to a : X_{\eta_b}(\omega), X_{\eta_b - 1}(\omega), \ldots, X_a(\omega).$$

we will denote the connected path by

$$\omega_{a\to c} := \omega_{a\to b} * \omega_{b\to c},$$

clearly

$$P(\omega_{a\to c}) = P(\omega_{a\to b})P(\omega_{b\to c}).$$

Now with these paths we construct the path to the neighbor level m_k that serves for estimation. Take the previously fixed path $\gamma_{a\to b}$ to the last visiting point b, then take the reversed path $\overline{\omega}_{b\to a}$. The next step is to go to a $c' \in m_0$ point along the path $\gamma_{a\to c'}$ for which $c \sim c'$ holds, because of the periodicity. Finally, translate the starting point of the path $\overline{\omega}_{c\to b}$ to $c' \in m_0$. Hence we arrive to the level line m_k, see *Figure 1*

Therefore for the estimation path we have

$$\gamma_{a,b} * \overline{\omega}_{b\to a} * \gamma_{a,c'} * \overline{\omega}_{c\to b}$$

The last term is because of the periodic environment

$$P(\overline{\omega}_{c\to b}) = P(\overline{\omega}_{c'\to b''})$$

where

$$b \sim b'' \sim b' \qquad c \sim c'$$

The estimation follows from the reversibility condition

$$\begin{aligned} P(\gamma_{a,b} * \overline{\omega}_{b\to a} * \gamma_{a,c'} * \overline{\omega}_{c\to b}) &= P(\gamma_{a,b})e^{u(b)-u(a)}P(\omega_{a\to b})P(\gamma_{a,c'})e^{u(c)-u(b)}P(\omega_{b\to c}), \\ &= e^{u(c)-u(a)}P(\gamma_{q,b})P(\gamma_{a,c'})P(\omega_{a\to c}) \qquad (3) \end{aligned}$$

where the exponential estimation came from the definition of potential. Now the desired result is just one step away.

$$P(\omega_{a\to c}) \le e^{u(a)-u(c)}C_\gamma^2 P(\overline{\omega}_{c\to a})$$

where C_γ^2 is a constant, depending on γ. Since $c \in m_{-k} \Rightarrow u(a) < u(c)$, and if we increase k, then $u(a) - u(c)$ decreases approximately linearly. Therefore we can push the level line m_{-k} so far that

$$P(\omega_{a\to c}) < qP(\overline{\omega}_{c\to a}),$$

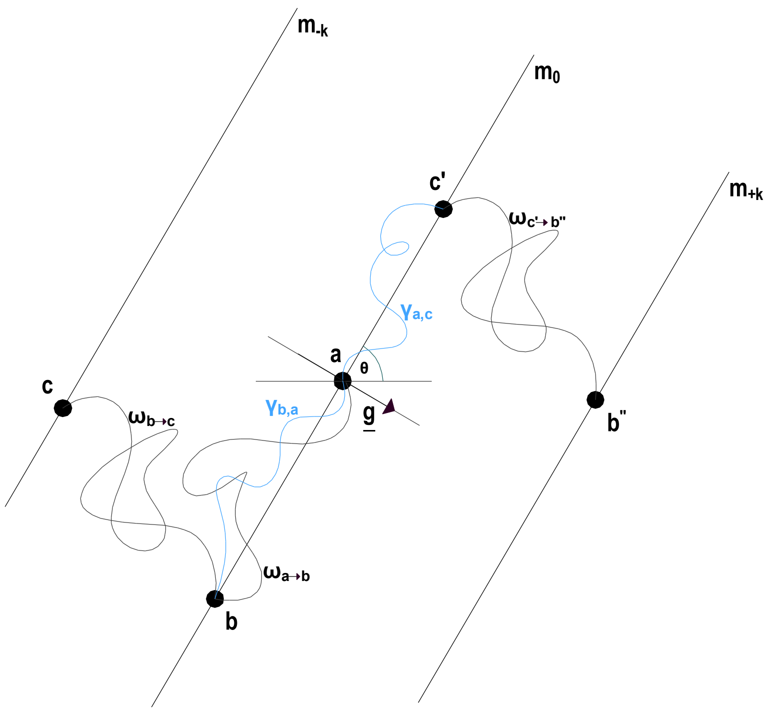

Figure 1: Construction of the estimation

with $q < 1$, independent from ω.Since the the the lines m_k determine independent areas, our desired result yields

$$P^{-k} < P^{+k}.$$

We need to highlight that this construction given above also works if the points b and c are different than above. Clearly the estimation will produce different result, but the procedure is applicable and ends up with the same relation.

Only the case $\tan\theta \in \mathbb{R}\backslash\mathbb{Q}$ left to be discussed. The whole idea is the same, but there are some key differences need to be thought. First of all, how to determine the level lines $l_g^{\pm k}$. Clearly on the line g_o there will not be any point $(x, y) \in \mathbb{Z}^2$, except the origin $(0, 0)$. Therefore the same can be told about l_g^0. However the gradient of the environment $\underline{g}$ can be approximated arbitrarily close, that is

$$\forall\varepsilon > 0, \quad \exists\underline{g}^a, \quad \text{such that, for} \quad \forall\underline{y} \in \mathbb{Z}^2, \quad \langle\underline{g}^a, \underline{y}\rangle \to \langle\underline{g}, \underline{y}\rangle,$$

let us denote these approximation vectors by $\underline{g}^a$. Now the same construction has to be built up on this $\underline{g}^a$. Therefore we get level lines, which approximates the 'original ' $l_g^{\pm k}$. Say the error term of the approximation is ε, arbitrarily given, and gives the potential differences. Then along the line l_g^0 fix a rectangular area with sides $K + K$ and $rK + rK$, so that the sides $rK + rK$ are parallel to l_g^0 and where r is the number of points with possible potentials in this area. Clearly the upper bound for the real and approximated potential deviation is $r\varepsilon$, which can be made arbitrarily small as well.

The only question needs to be addressed is that what is the possibility that the random walker 'escape' on the $K+K$ side of this rectangle. The answer is it has exponentially small probability, since Large Deviation Principle holds, and the number of probability measures are finite,because of the periodic environment. Indeed it can be made so by letting $K \to \infty$.

□

5 Conclusion

As mentioned in the previous Section the *Theorem 7* can be proved along the same argument as given in $d = 2$ only the appropriate modifications need to be done, e.g. the level lines have to be replaced be hyperplane in $\mathbb{Z}$, and the random walker's position is described by a $2d + 1$-dimensional simplex, i.e. we have $2d + 1$ vertices, where $+1$ is because we allow $p_x(0) \neq 0$.

This paper could be improved in deeper study of the finding out fine details about the relationship between the asymptotic velocity and gradient of the environment. We conjecture that with some certain conditions $\langle \underline{g}, \underline{\nu} \rangle \to 0$, i.e. the angle of these vectors can be close to rectangle. This investigation can be supported with further simulations.

In wider scale, in case of the general RWRE a still open problem is whether the asymptotic velocity is always an almost sure constant, recently this is the core subject of some research paper, see [1] and a lot of work has been done so far in the slowdown estimates. Mainly the hard nut is to give results in *annealed law* context. So this topic will have a lot to offer in the forthcoming years.

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