Diffusive behaviour of the myopic (or 'true') self-avoiding walk in three or more dimensions

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Outline of the talk

joint work with

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1. Introducion and model
2. Behaviour, conjectures
3. Environment process, stationary measure
4. Main results
5. Gaussian Hilbert spaces
6. Kipnis – Varadhan technology, sector conditions

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Myopic (or ‘true’) self-avoiding walk (MSAW)

$X(t)$ nearest neighbour random walk on $\mathbb{Z}^d$

local time (occupation time measure) with initialization:

$$l(t,x) := l(0,x) + |\{s \in [0,t] : X(s) = x\}|$$

Rate function:

$$w : \mathbb{R} \to (0,\infty) \qquad \inf_u w(u) = \gamma > 0$$

$$r(u) = \frac{w(u) - w(-u)}{2} \quad \text{increasing,} \qquad s(u) = \frac{w(u) + w(-u)}{2}$$

Jump rates:

$$\mathbf{P}\left(X(t+\mathrm{d}t) = y \,\middle|\, \text{past}, X(t) = x\right) = 1\!\!1_{\{|y-x|=1\}} w(l(t,x) - l(t,y))\,\mathrm{d}t$$

The walker is pushed by the discrete negative gradient of its own local time to less visited areas.

History, conjectures

MSAW introduced by Amit, Parisi, Peliti in 1983 in discrete time

Conjectures based on renormalization groups arguments:

- $d = 1$: $X(t) \sim t^{2/3}$; intricate, non-Gaussian scaling limit;
- $d = 2$: $X(t) \sim t^{1/2}(\log t)^{\xi}$; Gaussian scaling limit, $\xi = ?$;
- $d \geq 3$: $X(t) \sim t^{1/2}$; Gaussian scaling limit.

The physics literature:

- Amit, Parisi, Peliti (1983)
- Obukhov, Peliti (1983)
- Peliti, Pietronero (1987)

Continuous space analogue:
self-repellent Brownian polymer

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Rigorous results

$d = 1$

- Limit theorems for versions of MSAW:
  B. Tóth (1995), B. Tóth and B. V. (2009)

$$\frac{X(At)}{A^{2/3}} \Longrightarrow \mathcal{X}(t)$$

- Construction of scaling limit process $\mathcal{X}(t)$:
  B. Tóth, W. Werner (1998)
  $\mathcal{X}(t)$: true self-repelling motion (use of Brownian web)
- Super-diffusive bounds for the self-repellent Brownian polymer:
  P. Tarrès, B. Tóth, B. Valkó (2009)

$$ct^{5/4} \leq \mathbf{E}\left(X(t)^2\right) \leq Ct^{3/2}$$

Rigorous results

$d = 2$

- Super-diffusive bounds for the self-repellent Brownian polymer: B. Tóth, B. Valkó (2010)

$$ct \log \log t \le \mathbf{E}\left(X(t)^2\right) \le Ct \log t$$

- Expected order:

$$\mathbf{E}\left(X(t)^2\right) \sim t\sqrt{\log t}$$

$d \ge 3$

- Diffusive bounds and central limit theorem: I. Horváth, B. Tóth, B. Vető (2010)

$$\frac{X(t)}{\sqrt{t}} \Longrightarrow N(0, \sigma^2)$$

Environment as seen by the walker and its generator

$X(t)$: random walk in a dynamically changing random environment

Environment as seen from the position of the random walker:

$$\eta(t) = (\eta(t,x))_{x\in\mathbb{Z}^d}, \qquad \eta(t,x) = l(t, X(t)+x)$$

Markov process on the function space

$$\Omega := \{\omega = (\omega(x))_{x\in\mathbb{Z}^d} : \omega(x) \in \mathbb{R} \text{ with slow increase at infinity}\}.$$

Infinitesimal generator:

$$Gf(\omega) = \partial f(\omega) + \sum_{|e|=1} w(\omega(0)-\omega(e))(f(\tau_e\omega) - f(\omega))$$

where

$$\partial f(\omega) = \frac{\partial f}{\partial\omega(0)}(\omega) \quad \text{and} \quad \tau_e\omega(x) = \omega(x+e).$$

Stationary measure: massless free Gaussian field

If $d \geq 3$, let

$$R : \mathbb{R} \to [0, \infty) \qquad R(u) := \int_0^u r(v)\,\mathrm{d}v$$

where r is the odd part of w. The measures defined on finite subsets $\Lambda \subset \mathbb{Z}^d$ by

$$\mathrm{d}\pi\left(\omega_\Lambda \mid \omega_{\mathbb{Z}^d\setminus\Lambda}\right)$$
$$= Z_\Lambda^{-1} \exp\left\{ -\frac{1}{2} \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} R(\omega(x)-\omega(y)) - \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ |x-y|=1}} R(\omega(x)-\omega(y)) \right\} \mathrm{d}\omega_\Lambda$$

extend to a Gibbs measure on $\mathbb{Z}^d$. If $r(u) = u$, then this is the *massless free Gaussian field* with $\int_\Omega \omega(x)\,\mathrm{d}\pi(\omega) = 0$ and $\int_\Omega \omega(x)\omega(y)\,\mathrm{d}\pi(\omega) = (-\Delta)^{-1}_{xy} = C(y-x)$.

Stationary distribution and law of large numbers

Proposition

The probability measure π *is stationary and ergodic for the Markov process* $\eta(t)$ *on* Ω.

Corollary

For π *almost all initial profile* $l(0,\cdot)$,

$$\frac{X(t)}{t} \to 0$$

holds.

Central limit theorem

Theorem (I. Horváth, B. Tóth, B. V., 2010)

- *For any* $e \in \mathbb{R}^d$ *with* $|e| = 1$,

$$\gamma \leq \liminf_{t\to\infty} t^{-1}\mathbf{E}\left((e \cdot X(t))^2\right) \leq \limsup_{t\to\infty} t^{-1}\mathbf{E}\left((e \cdot X(t))^2\right) < \infty.$$

- *If* $r(u) = u$ *and* $s(u) = s_4 u^4 + s_2 u^2 + s_0$, *then the matrix of asymptotic covariances*

$$\sigma_{kl}^2 = \lim_{t\to\infty} t^{-1}\mathbf{E}\left(X_k(t) X_l(t)\right)$$

exists and it is non-degenerate. The finite dimensional distributions of

$$X_N(t) = N^{-1/2} X(Nt)$$

converge to those of a d *dimensional Brownian motion with covariance matrix* σ^2.

The Gaussian Hilbert space

The infinitesimal generator G of $\eta(t)$ acts on the graded Hilbert space (Fock space)

$$L^2(\Omega,\pi)=\bigoplus_{n=0}^{\infty}\mathcal{H}_n$$

with the Gaussian measure π.

To each $x\in\mathbb{Z}^d$, a Gaussian variable $\omega(x)$ is associated.

$\oplus_{k=0}^{n}\mathcal{H}_k$ contains the polynomials of the $\omega(x)$'s of maximal degree n.

$\mathcal{H}_n$ is orthogonalized to $\mathcal{H}_0,\dots,\mathcal{H}_{n-1}$ by the Gramm–Schmidt process.

$:\omega(x_1)\dots\omega(x_n):\in\mathcal{H}_n$ denotes the monomial $\omega(x_1)\dots\omega(x_n)$ orthogonalized to subspaces of lower indices, called a *Wick monomial*.

Wick polynomials are Hermite polynomials of an infinite number of variables.

Generator on the Gaussian space

$$G = -S + A = \frac{1}{2} \sum_{|e|=1} \nabla_{-e} s(a_e^* + a_e) \nabla_e + \sum_{|e|=1} (\nabla_{-e} a_e - a_e^* \nabla_{-e})$$

where S is the self-adjoint part, A is the skew self-adjoint part of G;

$$\nabla_e f(\omega) = f(\tau_e \omega) - f(\omega)$$

$$a_e^* :\omega(x_1) \dots \omega(x_n): = :(\omega(0) - \omega(e))\omega(x_1) \dots \omega(x_n):$$

$$a_e :\omega(x_1) \dots \omega(x_n): = \sum_{m=1}^{n} (C(x_m + e) - C(x_m)) :\omega(x_1) \dots \cancel{\omega(x_m)} \dots \omega(x_n):$$

A remark:

$$\Delta := \sum_{|e|=1} \nabla_e = -\sum_{k=1}^{d} \nabla_{e_k} \nabla_{e_k}^* .$$

Then

$$\left\| |\Delta|^{-1/2} \nabla_e \right\| \le 1.$$

Idea of proof

The displacement can be decomposed as

$$X(t) = M(t) + \int_0^t \varphi(\eta(s))\,\mathrm{d}s$$

where

- $M(t)$ is a martingale with stationary and ergodic increments,
- $\varphi_l(\omega) = w(\omega(0) - \omega(e_l)) - w(\omega(0) - \omega(-e_l))$ conditional speed.

Questions:

- Diffusive behaviour and central limit theorem for $M(t)$ is clear.
- Conditions of central limit theorem for the integral $\longrightarrow$ Kipnis – Varadhan theory
- For the diffusive lower bound, (partial) decorrelation is needed.
- Diffusive upper bound with Brascamp – Lieb inequality

Kipnis–Varadhan theory

General setup: $\eta(t)$ is a stationary and ergodic Markov process on the state space (Ω, π).

G is the infinitesimal generator of $\eta(t)$ acting on $L^2(\Omega, \pi)$.
Notation: $S := -\frac{1}{2}(G + G^*)$ and $A := \frac{1}{2}(G - G^*)$.
$f \in L^2(\Omega, \pi)$ with $\int_\Omega f \, \mathrm{d}\pi = 0$.

Question: sufficient condition for central limit theorem for

$$Y_N(t) := \frac{1}{\sqrt{N}} \int_0^{Nt} f(\eta(s)) \, \mathrm{d}s.$$

Sufficient conditions

- C. Kipnis, S. R. S. Varadhan, 1986 (reversible);
- B. Tóth, 1986 (non-reversible, discrete time);
- S. V. S. Varadhan, 1996: *(strong) sector condition*: diffusive bound on f and

$$\|S^{-1/2}AS^{-1/2}\| < \infty;$$

- S. Sethuraman, S. R. S. Varadhan, H-T. Yau, 2000: *graded/weak sector condition*: diffusive bound on f and $L^2(\Omega,\pi) = \oplus_{n=0}^{\infty}\mathcal{H}_n$ and
  - $S = \sum_n S_n$ with $S_n : \mathcal{H}_n \to \mathcal{H}_n$ and
  - $A = \sum_n A_{n+} + A_{n-}$ with $A_\pm : \mathcal{H}_n \to \mathcal{H}_{n\pm1}$

$$\left\| S^{-1/2}AS^{-1/2} \restriction_{\mathcal{H}_n} \right\| \le Cn^\gamma \quad \text{with } \gamma < 1.$$

Checking the graded sector condition

$$G = \frac{1}{2}\sum_{|e|=1} \nabla_{-e} s(a_e^* + a_e)\nabla_e + \sum_{|e|=1} (\nabla_{-e} a_e - a_e^* \nabla_{-e})$$

Heuristic meaning of sector conditions: the self-adjoint part S 'dominates' the skew self-adjoint part A

Generalization: $\Delta = \sum_{|e|=1} \nabla_e \leq S$ already 'dominates' A and the rest of S. In our case, if $d \geq 3$

$$\left\| |\Delta|^{-1/2} a_e^* \nabla_e |\Delta|^{-1/2} \restriction_{\mathcal{H}_n} \right\| \leq \left\| |\Delta|^{-1/2} a_e^* \restriction_{\mathcal{H}_n} \right\| \leq C n^{1/2}.$$

Enhancement of the generalized graded sector condition:

$\| |\Delta|^{-1/2} S |\Delta|^{-1/2} \restriction_{\mathcal{H}_n} \| \leq C n^2$ is enough instead of $\leq Cn$.

$$\left\| |\Delta|^{-1/2} \nabla_{-e} s(a_e^* + a_e) \nabla_e |\Delta|^{-1/2} \restriction_{\mathcal{H}_n} \right\| \leq \| s(a_e^* + a_e) \restriction_{\mathcal{H}_n} \| \leq C n^{\deg s/2}.$$

The end

Thank you for your attention!