The myopic or 'true' self-avoiding random walk

Illés Horváth, Bálint Tóth and Bálint Vető (pollux@math.bme.hu, balint@math.bme.hu, vetob@math.bme.hu)

Department of Stochastics, Institute of Mathematics, Budapest University of Technology and Economics, Hungary

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Abstract

The myopic (or 'true') self-avoiding walk is a random motion in $\mathbb{Z}^d$ which is pushed locally in the direction of the negative gradient of its own local time. This transition rule defines a family of self-repelling random processes which have different asymptotic behaviour in different dimensions. We present several models in one and high dimensions ($d \geq 3$), and we prove limit theorems which describe the scaling properties and the limiting behaviour of these processes.

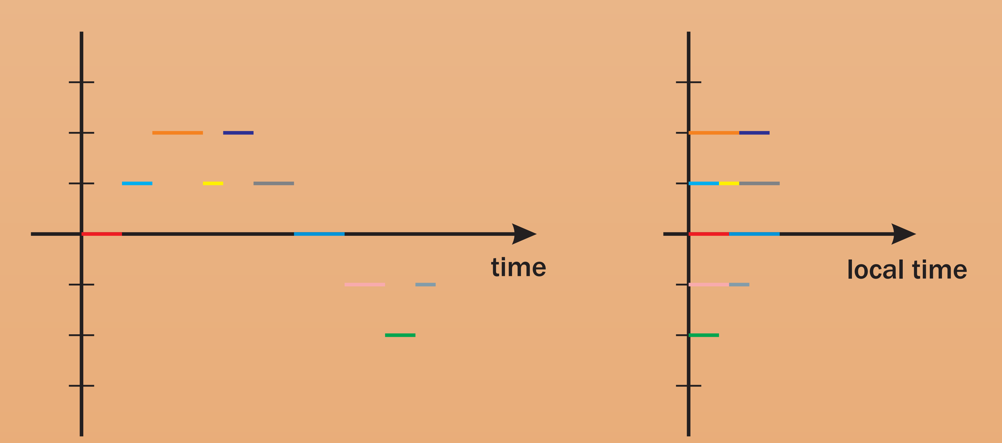
Introduction

The 'true' self-avoiding walk appeared first in the physics literature in 1983 as a natural random model of self-repellence, see [APP83]. The problem was originally introduced in discrete time, but here, we define it as a continuous time nearest neighbour jump process on $\mathbb{Z}^d$ in the following way.

Let $X(t)$, $t \in \mathbb{R}_+$ be the process to be defined, and we call

$$\ell(t,x) := \ell(0,x) + |\{0 \leq s \leq t : X(s) = x\}|$$

the occupation time measure (or local time) of $X(t)$ at $x \in \mathbb{Z}^d$ with some initial value $\ell(0,x)$.



Let $w : \mathbb{R} \to (0,\infty)$ a fixed smooth non-decreasing 'rate function'. Then the law of $X(t)$ is given by the jump rates

$$(1)\quad \mathbf{P}\left(X(t+\mathrm{d}t) = y \;\middle|\; \mathcal{F}_t, X(t) = x\right) = \mathbb{1}(|x-y| = 1)\, w(\ell(t,x) - \ell(t,y))\,\mathrm{d}t + o(\mathrm{d}t)$$

where $\mathcal{F}_t$ is the information which is known up to time t. Note that it includes all the values of $\ell(t,x)$, $x \in \mathbb{Z}^d$.

Dimension-dependent behaviour

Non-rigorous scaling and renormalization group arguments suggest the following conjecture on the dimension-dependent asymptotic scaling behaviour:

- $d = 1$: $X(t) \sim t^{2/3}$ with intricate non-Gaussian scaling limit,
- $d = 2$: $X(t) \sim t^{1/2}(\log t)^{\zeta}$ with Gaussian scaling limit (ζ unknown),
- $d \geq 3$: $X(t) \sim t^{1/2}$ with Gaussian scaling limit.

This conjecture was set up by the authors of [APP83] and confirmed by subsequent rigorous results. Our contribution in one dimension is that we identified the limit behaviour of the walk formulated above, and we also treated another variant. We remark that the two dimensional case was investigated by B. T. and B. Valkó. We also prove the central limit theorem for the high dimensional case of the myopic self-avoiding walk and the self-repelling Brownian polymer.

On this poster, we formulate our results in one dimension and the limit theorems in three or more dimensions.

One dimension

As indicated above, the proper scaling of the one dimensional myopic self-avoiding walk is $t^{2/3}$ after time t. More precisely, in [TV10], we prove the following theorems for the random walk defined by (1) in late random times.

The inverse local time of the model is

$$T_{j,r} := \inf\{t \geq 0 : \ell(t,j) \geq r\}$$

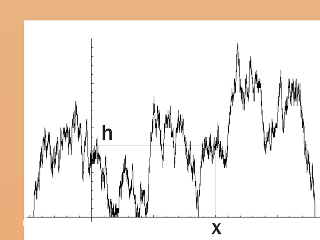
for $j \in \mathbb{Z}$ and $r \in \mathbb{R}_+$. The sequence of local times at $T_{j,r}$ is

$$(2)\quad \Lambda_{j,r}(k) := \ell(T_{j,r}, k) \quad \text{for all} \quad k \in \mathbb{Z}$$

With proper scaling, we describe the limit behaviour of this process.

Theorem (B. T., B. V., 2009). *Let $x \geq 0$. There is a $\sigma \in (0,\infty)$ such that*

$$\left(\frac{\Lambda_{Ax,\sqrt{A}\sigma h}(Ay)}{\sigma\sqrt{A}}\right)_{y \in \mathbb{R}} \Longrightarrow$$



as $A \to \infty$ where the limit process is the absolute value of a two-sided standard Brownian motion $|W_y|$ started at $|W_x| = h$ between 0 and x backwards which is absorbed at the first hitting time of 0 on $\{y > x\}$ forward and on $\{y < 0\}$ backward.

Theorem (B. T., B. V., 2009). *The rescaled displacement $X(At)/A^{2/3}$ converges to the true self-repelling motion defined in [TW98] (both stopped at an independent exponential random time).*

One dimension with directed edge repulsion

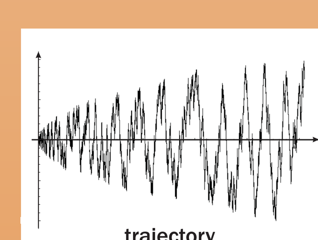
A discrete-time model is analyzed in [TV08]. We define the local time of oriented edges. The transition probabilities depend on the local times of the edges pointing out of the current position similarly to (1). A surprisingly different limit behaviour is obtained:

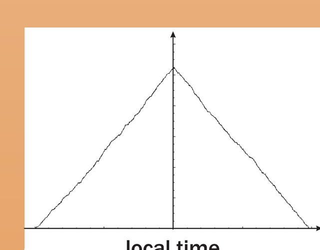
Theorem (B. T., B. V., 2008). *1.*

$$A^{-1/2}X(At) \Longrightarrow \mathrm{UNI}(-\sqrt{t}, \sqrt{t})$$

as $A \to \infty$ with no continuous limit process.

2. The rescaled local time process analogous to (2) converges to a deterministic triangular shape function.





Ray–Knight approach

The basic idea of the proofs appeared first in [T95]. The differences of local times on adjacent vertices/edges are auxiliary Markov chains which are independent. These Markov chains are close to their stationary distribution if the local time is high at the corresponding site/bond. In the directed edge repulsion case, the expected value in stationarity is non-zero, hence LLN holds, and the limit is deterministic. In the site repulsion case, the expected value is 0, and the CLT for simple random walks gives the Brownian limit.

Dimensions three and higher

In the high dimensional case, the stationary measure is identified first, and the following results are valid in the stationary regime, i.e. when the local times are initialized according to the stationary measure of the walk.

The stationary measure is constructed as a Gibbs measure, but can be formulated loosely as

$$\mathrm{d}\pi(\omega) = Z^{-1} \exp\left\{-\frac{1}{2} \sum_{x,y \in \mathbb{Z}^d, |x-y|=1} R(\omega(x) - \omega(y))\right\}$$

where

$$R(u) := \int_0^u r(v)\,\mathrm{d}v, \quad r(u) = \frac{w(u) - w(-u)}{2}$$

and Z is a normalizing constant.

In the stationary regime, the displacement scales like $t^{1/2}$, and the finite dimensional distributions of the rescaled displacement process converge to those of a d dimensional Brownian motion with covariance matrix σ^2. For precise assumptions regarding the rate function w, see [HTV10].

Theorem (I. H., B. T., B. V., 2010). *1. For a wide class of rate functions w (including polynomials),*

$$0 < \gamma \leq \inf_{\substack{e \in \mathbb{R}^d \\ |e|=1}} \varliminf_{t\to\infty} t^{-1}\mathbf{E}\left((e \cdot X(t))^2\right),$$

$$\sup_{\substack{e \in \mathbb{R}^d \\ |e|=1}} \varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left((e \cdot X(t))^2\right) < \infty.$$

2. For a more restricted class of rate functions, the finite dimensional distributions of the rescaled displacement process

$$X_N(t) := N^{-1/2}X(Nt)$$

converge to those of a d dimensional Brownian motion with some covariance matrix σ^2.

Idea of proof

The main idea is to consider the local time profile as seen from the position of the random walker

$$\eta(t) := (\eta(t,x))_{x \in \mathbb{Z}^d} \quad \text{with} \quad \eta(t,x) := \ell(t, X(t) + x).$$

The displacement can be written as

$$(3)\quad X(t) = M(t) + \int_0^t \varphi(\eta(s))\,\mathrm{d}s$$

where $M(t)$ is a martingale and φ is the infinitesimal conditional speed, i.e. $\varphi_k(\omega) = w(\omega(0) - \omega(e_k)) - w(\omega(0) - \omega(-e_k))$ where e_k is the kth unit vector and $k = 1, \dots, d$. This is the setup of [KV86]. The martingale part in (3) behaves diffusively, but the integral requires more effort, since the process $\eta(t)$ is not reversible. We check the *graded sector condition*, see [SVY00]. The generator of $\eta(t)$ is

$$G = -\gamma\Delta - \sum_{|e|=1} \nabla_{-e} w_{\text{symm}}(a_e^* + a_e)\nabla_e + \sum_{|e|=1} \nabla_{-e}a_e - a_e^*\nabla_{-e},$$

acting on a Gaussian Hilbert space.

Continuous space variant: the self-repelling Brownian polymer model

The self-repelling Brownian polymer model, which is a continuous space variant of the true self-avoiding walk, was initiated by J. Norris, C. Rogers and D. Williams in 1987 in [NRW87].

$X(t)$ is a diffusion process in $\mathbb{R}^d$. The occupation time measure is defined by

$$\ell(t,A) := \ell(0,A) + |\{s \in [0,t] : X(s) \in A\}|$$

where $\ell(0,A)$ is a random initialization sampled from the distibution of the *massless free Gaussian field* on $\mathbb{R}^d$. This is a stationary measure for the local time profile as seen from the position $X(t)$.

The evolution of this process is given in terms of the smeared-out local times as follows:

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \mathrm{grad}(V * \ell(t,\cdot))(X(t))\,\mathrm{d}t$$

where $V : \mathbb{R}^d \to \mathbb{R}^+$ is an approximate identity, e.g. $V(x) = e^{-|x|^2}$.

Theorem (I. H., B. T., B. V., 2009). *1. The finite dimensional marginal distributions of the rescaled process*

$$X_N(t) := \frac{X(Nt)}{\sigma\sqrt{N}}$$

converge to those of a standard d dimensional Brownian motion for some $0 < \sigma < \infty$. The convergence is meant in probability with respect to the starting state $\eta(0)$ sampled according to the stationary distribution.

2. For the limiting variance

$$1 \leq d^{-1} \lim_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) \leq 1 + \rho^2$$

holds with some $\rho < \infty$.

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