Models of the ‘true’ self-avoiding walk on $\mathbb{Z}$

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Introduction

joint work with Bálint Tóth

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$X(n)$ nearest neighbour random walk with $X(0) = 0$.
Local times on sites:

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$w : \mathbb{Z} \to \mathbb{R}_+$ almost arbitrary weight function, non-decreasing, e.g. $w(k) = e^{\beta k}$ with $\beta > 0$.

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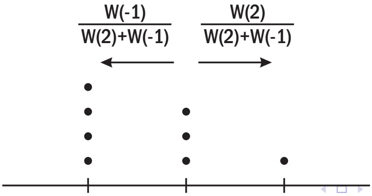

Questions: asymptotic behaviour

- Scaling limit of the local time

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$$A^{-\nu}X([At]) \Rightarrow \mathcal{X}(t)$$

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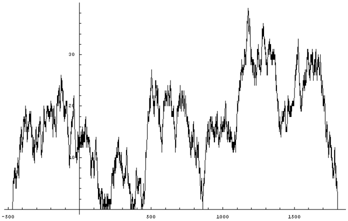

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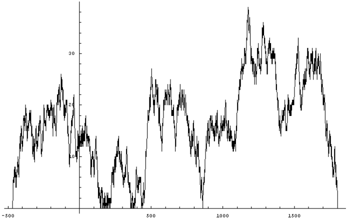

- $\mathcal{X}(t)$ (scaling limit): true self-repelling motion (Tóth – Werner, 1998)

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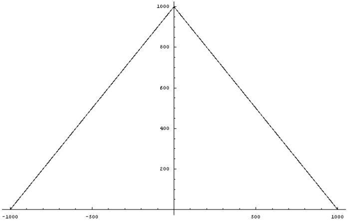

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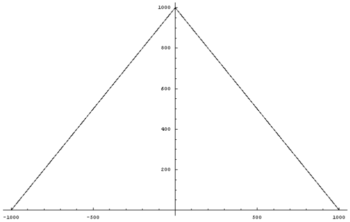

- $\mathcal{X}(t)$: uniform on $\left[-\sqrt{t}, \sqrt{t}\right]$ $\left(\frac{X(At)}{\sqrt{A}} \Rightarrow \mathcal{X}(t)\right)$
  no continuous limit process

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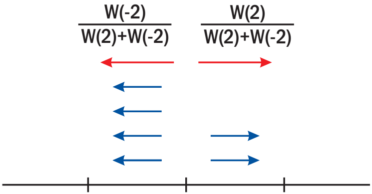

Auxiliary Markov-chains

$$\eta_{k,\pm}(n) = \mp(\ell^+(t(n),k) - \ell^-(t(n),k))$$

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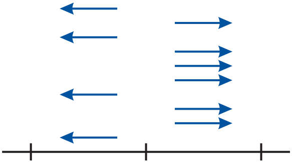

$t(4) = 10$ $\quad \eta_{k,-}(4) = 2$

$t(3) = 8$ $\quad \eta_{k,-}(3) = 2$

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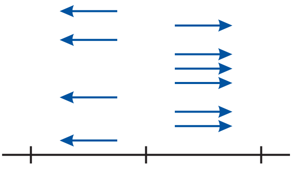

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$\eta_{k,\pm}$ are i.i.d. Markov-chains (if we choose either $+$ or $-$ for each $k \in \mathbb{Z}$)

Recursion on $\Lambda_{i,m}$

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$$\begin{aligned}
\Lambda_{i,m}(k+1) &= \Lambda_{i,m}(k) + \eta_{k+1,-}(\Lambda_{i,m}(k)) && \text{if} \quad k \geq i \\
\Lambda_{i,m}(k-1) &= \Lambda_{i,m}(k) + \eta_{k,+}(\Lambda_{i,m}(k)-1)+1 && \text{if} \quad 0 < k \leq i \\
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$\Lambda_{i,m}$ is a random walk.
The step distribution depends on the position.

Stationary distribution of $\eta_{k,\pm}$

Lemma

The unique stationary distribution of the Markov-chains $\eta_{k,\pm}$ is defined by

$$\rho(k) = \rho(-k-1) = Z^{-1} \prod_{l=1}^{k} \frac{w(-l)}{w(l)} \quad \textit{where} \quad Z = 2 \sum_{r=0}^{\infty} \prod_{l=1}^{r} \frac{w(-l)}{w(l)}$$

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There are constants $c_1 < \infty$ and $c_2 > 0$ such that

$$\sum_{y \in \mathbb{Z}} |P^n(0,y) - \rho(y)| < c_1 e^{-c_2 n}$$

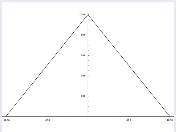
where $P^n(x,y) = \mathbf{P}\left(\eta_{k,\pm}(n) = y \,\middle|\, \eta_{k,\pm}(0) = x\right)$.

Limit theorem for the local time process

Theorem (B. Tóth, B. V., 2008)

Let $x \in \mathbb{R}$ *and* $h \in \mathbb{R}_+$ *fixed. Then*

$$A^{-1}\Lambda_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(\lfloor Ay \rfloor) \Rightarrow$$



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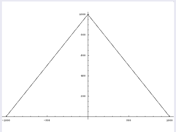

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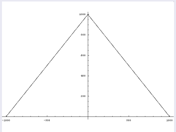

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It can be shown that if $\Lambda_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}$ is at most $\sqrt{A}$, then it reaches 0 in $o(A)$ time with large probability.

Limit theorem for the position of the random walker

Conjecture

$$\frac{X([At])}{\sqrt{A}} \Rightarrow \mathrm{UNI}[-\sqrt{t}, \sqrt{t}]$$

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Theorem (B. Tóth, B. V., 2008)

Let $\theta_{s/A}$ *be independent of the walk* X *with geometric distribution*

$$\mathbf{P}\left(\theta_{s/A} = n\right) = \left(1 - e^{-s/A}\right)\left(e^{-s/A}\right)^n.$$

Then

$$\frac{X(\theta_{s/A})}{\sqrt{A}} \Rightarrow Y$$

where the density of Y *is* $x \mapsto s\int_0^\infty e^{-st}\frac{1}{2\sqrt{t}}\mathbb{1}(|x| \leq \sqrt{t})\,\mathrm{d}t$.

Simulation results

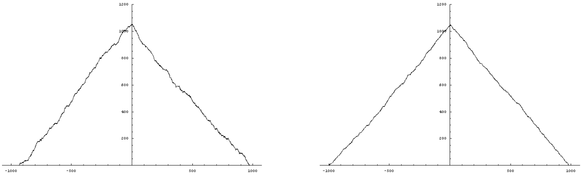

Figure: Local time process $\Lambda_{100,800}$ with $w(k) = 2^k$ and $w(k) = 10^k$

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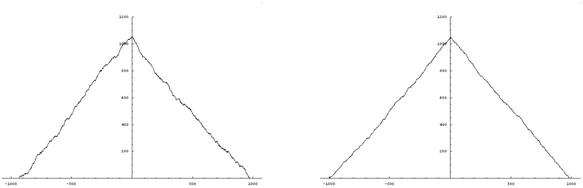

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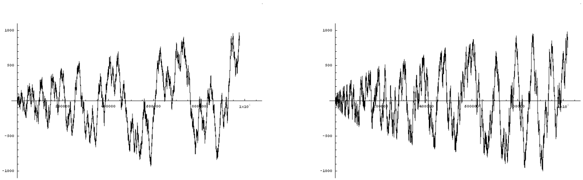

Figure: Trajectories of $X(n)$ with $w(k) = 2^k$ and $w(k) = 10^k$

Continuous time, site repulsion

$$\ell(t,j) := |\{0 \le s < t : X(s) = j\}|$$

$$\mathbf{P}\left(X(t+\mathrm{d}t) = j \pm 1 \mid \text{past}, X(t) = j\right) = w(\ell(t,j) - \ell(t,j \pm 1))\,\mathrm{d}t$$

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Infinitesimal generator of the auxiliary Markov-processes $\eta_{k,\pm}$:

$$(Kf)(x) = -f'(x) + \int_{\mathbb{R}} r(u,v)(f(v) - f(u))\,\mathrm{d}v$$

where $r(u,v) = \mathbb{1}(v > u)w(-u)\exp\left(-\int_u^v w(s)\,\mathrm{d}s\right) w(v)$

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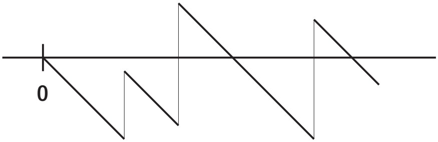

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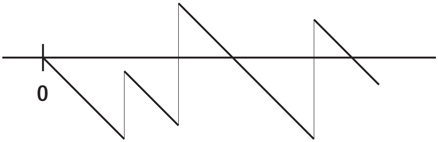

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where $P^t(x, \mathrm{d}y) = \mathbf{P}\left(\eta_{k,\pm}(t) \in \mathrm{d}y \;\middle|\; \eta_{k,\pm}(0) = x\right)$*.*

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The rate of merge provided that $\eta_1 = x_1$ and $\eta_2 = x_2$:

$$w(-x_1 \vee x_2) \exp\left(-\int_{x_1 \wedge x_2}^{x_1 \vee x_2} w(z)\,\mathrm{d}z\right) \geq w(-b) \exp\left(-\int_{-\infty}^{b} w(z)\,\mathrm{d}z\right)$$

if $x_1 \vee x_2 < b$.

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$$\mathbf{P}(T > t) \leq \mathbf{P}\left(\vartheta_t < \frac{t}{2}\right) + \mathbf{P}\left(T > t \mid \vartheta_t \geq \frac{t}{2}\right)$$

where $\vartheta_t = |\{0 \leq s \leq t : \eta_1(s) \vee \eta_2(s) < b\}|$.

Open questions

- rigorous proof for the original model (discrete time, site repulsion)
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  generalization of Kipnis – Varadhan-theorem for the non-revesible case (central limit theorem for additive functionals of Markov-processes)
  Markov-process: environment seen from the position of the walker
  joint work with I. Horváth and B. Tóth

Thank you for the attention!