Tracy–Widom asymptotics for q-TASEP

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Introduction

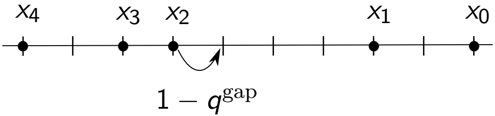
joint work with Patrik Ferrari

Outline:

- Introduction
- Macroscopic behaviour
- Tracy–Widom limit under time$^{1/3}$ scaling
- Main steps of the proof

q-TASEP

q-TASEP (totally asymmetric simple exclusion process): continuous time interacting particle system on $\mathbb{Z}$ for $q \in [0,1)$



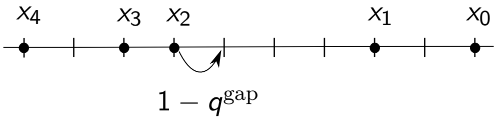
Configurations: $\mathbf{x} = (x_N : N \in \mathbb{Z} \text{ or } N \in \mathbb{N})$ where $x_N < x_{N-1}$ for all N

Dynamics: particle x_N jumps to the right by 1 with rate $1 - q^{x_{N-1} - x_N - 1}$, i.e. the infinitesimal generator is

$$(Lf)(\mathbf{x}) = \sum_N \left(1 - q^{x_{N-1} - x_N - 1}\right) \left(f\left(\mathbf{x}^N\right) - f(\mathbf{x})\right)$$

where $\mathbf{x}^N$ is the configuration where x_N is increased by one

q-TASEP



Infinitesimal generator

$$(Lf)(\mathbf{x}) = \sum_N \left(1 - q^{x_{N-1} - x_N - 1}\right) \left(f\left(\mathbf{x}^N\right) - f(\mathbf{x})\right)$$

where $\mathbf{x}^N$ is the configuration where x_N is increased by one

Observation: $x_N < x_{N-1}$ is preserved by the dynamics

$q = 0$ is TASEP

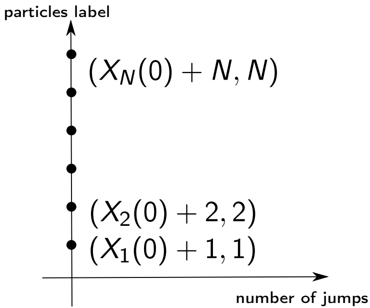
Configuration at time τ: $\mathbf{X}(\tau) = (X_N(\tau) : N \in \mathbb{Z}$ or $N \in \mathbb{N})$

Step initial condition: $X_N(0) = -N$ for $N = 1, 2, \ldots$

Macroscopic evolution of the vector

$$(X_N(\tau) + N, N)$$

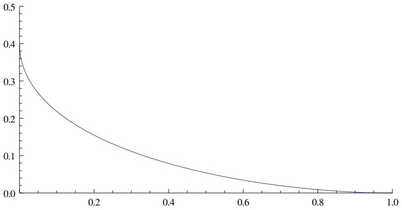
initially



after normalizing by τ, i.e.

$$\left(\frac{X_N(\tau) + N}{\tau}, \frac{N}{\tau}\right)$$

as $\tau \to \infty$



Macroscopic behaviour

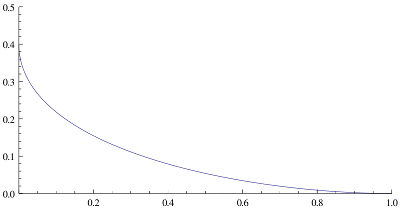
Proposition (LLN)

Let $\kappa > 1/(1-q)$ be fixed. Then

$$\frac{X_N(\tau = \kappa N)}{N} \to f - 1$$

as $N \to \infty$ where f depends on κ implicitly. Equivalently

$$\left(\frac{X_N(\tau) + N}{\tau}, \frac{N}{\tau}\right) \to \left(\frac{f}{\kappa}, \frac{1}{\kappa}\right).$$



There are explicit parametric formulas $\kappa(\theta), f(\theta)$.
The figure shows the parametric plot of $(f/\kappa, 1/\kappa)$.
The touching point on the vertical axis is at $1-q$.

Results

Let us write

$$X_N(\kappa N) = (f-1)N + \frac{\chi^{1/3}}{\log q}\xi_N N^{1/3}$$

where ξ_N is the rescaled particle position and $\chi > 0$ is a constant which depends on κ.

Theorem (P. Ferrari, B. V., 2013)

1. *If $\kappa \in (1/(1-q), \kappa_0)$ for some $\kappa_0 = \kappa_0(q)$, then*

$$\lim_{N\to\infty} \mathbf{P}(\xi_N < x) = F_{\mathrm{GUE}}(x)$$

where F_{GUE} is the GUE Tracy–Widom distribution function.

2. *This confirms the KPZ scaling theory conjecture on the variance of the particle current.*

History

q-TASEP

- Borodin, Corwin, 2011: first introduction of q-TASEP
- Borodin, Corwin, Ferrari, 2012: Fredholm determinant formula for the q-Laplace transform of the particle position in q-TASEP (suitable for asymptotics)

q-Boson particle system

- Sasamoto, Wadati, 1998: introduction of q-Boson particle system
- Borodin, Corwin, Sasamoto, 2012: duality of q-Boson particle system and q-TASEP, joint moment formulas for multiple particle positions (not Fredholm determinant)
- Borodin, Corwin, Petrov, Sasamoto, 2013 and Korhonen, Lee, 2013: analysis of q-Boson particle system with Bethe ansatz

Asymptotics

- Ferrari, V, 2013: asymptotics for q-TASEP
- Barraquand, 2014: upper bound on κ removed

Stationary q-TASEP

Stationary q-TASEP: i.i.d. q-geometrically distributed gaps between particles with paramter $\alpha \in [0,1)$, that is,

$$\mathbf{P}(\mathrm{gap} = k) = (\alpha; q)_\infty \frac{\alpha^k}{(q;q)_k} \quad \text{for } k = 0, 1, 2, \ldots$$

where $(a;q)_k = (1-a)(1-aq)\ldots(1-aq^{k-1})$ is the q-Pochhammer symbol

Proposition

For stationary q-TASEP with paramter α, the particle density ρ and the corresponding particle current $j(\rho)$ are explicit:

$$\rho = \frac{\log q}{\log q + \log(1-q) + \Psi_q(\log_q \alpha)}, \quad j(\rho) = \alpha\rho$$

where Ψ_q is the q-digamma function.

Hydrodynamic limit

Proposition

1. *The function*

$$\rho\left(t, \frac{f(\theta)-1}{\kappa(\theta)} t\right) = \frac{\log q}{\log q + \log(1-q) + \Psi_q(\theta)}$$

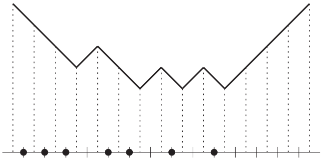
where the right-hand side is the stationary particle density with parameter $\alpha = q^\theta$ *solves the mass conservation PDE*

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x} j(\rho(t,x)) = 0$$

with initial condition $\rho(0,x) = \mathbb{1}(x<0)$ *where* $j(\rho)$ *is the particle current at density* ρ*.*

2. *Under the assumption of local stationary, the hydrodynamic limit exists and the density profile is as above. Further, the law of large numbers* $X_N(\tau = \kappa N)/N \to f - 1$ *holds.*

KPZ scaling theory conjecture



Height function corresponding to the particle system

$\eta_j = h_{j+1} - h_j$ height difference

Generator

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} c_{j,j+1}(\eta)(f(\eta^{j,j+1}) - f(\eta))$$

Stationary measure μ_ρ indexed by $\rho = \lim_{a\to\infty} \frac{1}{2a+1} \sum_{|j|\leq a} \eta_j$

Steady state current: $j(\rho) = \mu_\rho(c_{0,1}(\eta)(\eta_0 - \eta_1))$ and $\lambda(\rho) = -j''(\rho)$

Integrated covariance: $A(\rho) = \sum_{j\in\mathbb{Z}}(\mu_\rho(\eta_0\eta_j) - \mu_\rho(\eta_0)^2)$

Conjecture (KPZ scaling, Spohn, 2012)

If $\phi(y) = \sup_{|\rho|\leq 1}(y\rho - j(\rho))$ *and we set* $\rho = \phi'(y)$, *then*

$$\lim_{t\to\infty} \mathbf{P}\left(h(yt,t) - t\phi(y) \geq -(-\tfrac{1}{2}\lambda A^2)^{1/3} s t^{1/3}\right) = F_{\mathrm{GUE}}(s).$$

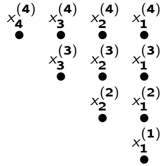
Finite time formula

Theorem (Borodin, Corwin, Ferrari, 2012)

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. *Then*

$$\mathbf{E}\left(\frac{1}{\left(\zeta q^{X_N(\tau)+N};q\right)_\infty}\right) = \det(\mathbb{1} - K_\zeta)_{L^2(C)}$$

where K_ζ *is an explicit trace class operator given by its integral kernel and* C *is a contour in the complex plane.*



Formula comes from the q-Whittaker 2d growth model (introduced in the study of Macdonald processes by Borodin, Corwin, 2011)

q-Whittaker 2d growth model: Markov process on the space of Gelfand–Tsetlin patterns

$$\left\{x_k^{(m)} : 1 \le k \le m \le N : x_{k+1}^{(m+1)} < x_k^{(m)} \le x_k^{(m+1)}\right\}$$

Particles $\{x_k^{(k)}, k = 1, 2, \dots\}$ follow q-TASEP

Convergence of the q-Laplace transform

$$X_N(\tau=\kappa N)=(f-1)N+\frac{\chi^{1/3}}{\log q}\xi_N N^{1/3}$$

and with $\zeta=-q^{-fN-\frac{\chi^{1/3}}{\log q}xN^{1/3}}$ for some $x\in\mathbb{R}$, the left-hand side of the finite time formula can be written as

$$\begin{aligned}\mathbf{E}\left(\frac{1}{\left(\zeta q^{X_N(\tau=\kappa N)+N};q\right)_\infty}\right)&=\mathbf{E}\left(\frac{1}{\left(-q^{\frac{\chi^{1/3}}{\log q}(\xi_N-x)N^{1/3}};q\right)_\infty}\right)\\&=\mathbf{E}\left(\frac{1}{\prod_{k=0}^\infty\left(1+q^{\frac{\chi^{1/3}}{\log q}(\xi_N-x)N^{1/3}+k}\right)}\right)\\&\to\mathbf{P}(\xi_N<x)\end{aligned}$$

as $N\to\infty$

Asymptotic analysis

Proposition

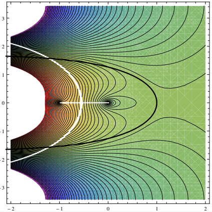
For $\zeta = -q^{-fN - \frac{\chi^{1/3}}{\log q} x N^{1/3}}$ *with some* $x \in \mathbb{R}$*, the Fredholm determinant converges*

$$\det(\mathbb{1} - K_\zeta)_{L^2(C)} \to \det(\mathbb{1} - K_{\mathrm{Ai},x})_{L^2(\mathbb{R}_+)} = F_{\mathrm{GUE}}(x)$$

where

$$K_{\mathrm{Ai},x}(u,v) = \int_0^\infty \mathrm{d}\lambda \, \mathrm{Ai}(u+x+\lambda)\, \mathrm{Ai}(v+x+\lambda)$$

is the shifted Airy kernel.



The proposition relies on the steep descent property of the black curve C for the function indicated with the colors.

The end

Thank you for your attention!