Central limit theorem for the Brownian polymer model

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Outline of the talk

joint work with

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Motivation

Introduction, result, conjectures

Environment process, stationary measure

Gaussian Hilbert space, operators, generator

Kipnis – Varadhan technology, sector condition

Myopic (or 'true') self-avoiding walk (TSAW)

D. Amit, G. Parisi, L. Peliti, 1983

in continuous time:
$X(t)$ nearest neighbour random walk on $\mathbb{Z}^d$

local time (occupation time measure) with initialization:
$l(t,x) := l(0,x) + |\{s \in [0,t] : X(s) = x\}|$

Jump rates:

$$\mathbf{P}\left(X(t+\mathrm{d}t) = y \,\middle|\, \text{past}, X(t) = x\right) = \mathbb{1}_{\{|y-x|=1\}} r(l(t,x) - l(t,y))\,\mathrm{d}t + o(\mathrm{d}t)$$

where $r : \mathbb{R} \to (0,\infty)$ increasing.

The walker is pushed by the discrete negative gradient of its own local time to less visited areas.

Self-repelling Brownian polymer model (SRBP)

J. Norris, C. Rogers, D. Williams, 1987
R. Durrett, C. Rogers, 1992

$X(t)$ diffusion process in $\mathbb{R}^d$

occupation time measure with initialization:
$l(t,A) := l(0,A) + |\{s \in [0,t] : X(s) \in A\}|$

$V : \mathbb{R}^d \to \mathbb{R}^+$ approximate identity, e.g. $V(x) = e^{-|x|^2}$
$F : \mathbb{R}^d \to \mathbb{R}^d$, $F(x) = -\operatorname{grad} V(x)$

Evolution:

$$X(t) = B(t) + \int_0^t \int_0^s F(X(s) - X(u))\,\mathrm{d}u\,\mathrm{d}s$$

or

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \left(\int_0^t F(X(t) - X(u))\,\mathrm{d}u\right)\mathrm{d}t$$

or

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \operatorname{grad}(V * l(t,\cdot))(X(t))\,\mathrm{d}t.$$

Earlier results, conjectures

dimension-dependent behaviour for both models

$d = 1$ $X(t) \sim t^{2/3}$ with difficult non-Gaussian scaling limit

- limit theorem for a version of 1d TSAW (B. Tóth, 1995)
- construction of the limit process (B. Tóth, W. Werner, 1998)
- another version of TSAW in 1d (B. Tóth, B. V., 2009)

$d = 2$ $X(t) \sim t^{1/2}(\log t)^{\xi}$ with Gaussian limit, $\xi =?$

- partial results (B. Valkó, 2009)

$d \geq 3$ $X(t) \sim t^{1/2}$ with Gaussian limit

- CLT for the SRBP (I. Horváth, B. Tóth, B. V., 2009)

Environment seen by the walker

$\xi(t) : \mathbb{R}^d \to \mathbb{R}$, $\xi(t) := \xi(0) + (V * l)(t)$ environment process (smeared out local time with initialization)

$$\xi(t,x) = \xi(0,x) + \int_0^t V(x - X(s))\,\mathrm{d}s.$$

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$\eta(t) : \mathbb{R}^d \to \mathbb{R}$, $\eta(t,x) := \xi(t, X(t) + x)$ environment as seen by the walker

$$\eta(t,x) = \eta(0,x) + \int_0^t V(X(t) + x - X(s))\,\mathrm{d}s.$$

$\eta(t)$ is a Markov process in some state space Ω (function space)

Stationary distribution exists (P. Tarrès, B. Tóth, B. Valkó, 2009 in 1d)

Stationary measure: massless free Gaussian field

Condition on V: positive type, i.e.

$$\widehat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} V(x)\, \mathrm{d}x \geq 0.$$

$\Omega := \{\omega : \mathbb{R}^d \to \mathbb{R}$ smooth with slow increase at $\infty\}$,

A random element $\omega \in \Omega$ is a Gaussian field, if $(\omega(x))_{x\in\mathbb{R}^d}$ are jointly Gaussian random variables.

Choose the distribution $\pi(\mathrm{d}\omega)$ in such a way that

$$\mathbf{E}_\pi(\omega(x)) = 0, \qquad \mathbf{E}_\pi(\omega(x)\omega(y)) = C(y-x)$$

with

$$C = (-\Delta)^{-1} V, \quad \text{more precisely} \quad \widehat{C}(p) = |p|^{-2}\widehat{V}(p).$$

This is the massless free Gaussian field smeared out by V. Note that it exists in 3 or more dimensions.

Main results

Theorem

$\eta(t)$ is a stationary and ergodic Markov process on (Ω, π).

Corollary

$$X(t)/t \to 0 \qquad a.s.$$

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Theorem (I. Horváth, B. Tóth, B. V., 2009)

- $\sigma := \lim_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right)$ *exists,*
- $d \leq \sigma^2 \leq d + \int_{\mathbb{R}^d} |p|^{-2}\widehat{V}(p)\,\mathrm{d}p < \infty$,
- $$\frac{X(Nt)}{\sigma\sqrt{N}} \Longrightarrow W(t)$$

  in the sense of finite dimensional marginals.

Basic idea: CLT for additive functionals of Markov processes
$\varphi : \Omega \to \mathbb{R}^d$, $\varphi(\omega) := -\operatorname{grad}\omega(0)$

$$X(t) = B(t) + \int_0^t \varphi(\eta(s))\,\mathrm{d}s.$$

Gaussian Hilbert space, an example

The space of interest is $L^2(\Omega, \pi)$.

Example

Instead of (Ω, π)*, consider* $\Omega_{ex} := \mathbb{R}$ *with*
$\pi_{ex}(\mathrm{d}x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x$.
There is an orthogonal decomposition

$$L^2(\Omega_{ex}, \pi_{ex}) = \oplus_{n=0}^{\infty} \mathcal{H}_n^{ex}$$

where $\mathcal{H}_n^{ex}$ *contains polynomials of degree* n*.*

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where $\mathcal{H}_n^{ex}$ *contains polynomials of degree* n*. These are the Hermite polynomials, which can be constructed with the Gramm–Schmidt orthogonalization.*

Similarly with infinitely many variables, the same procedure gives

$$L^2(\Omega, \pi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$$

where $\mathcal{H}_n$ is generated by the Wick polynomials of form $:\omega(x_1)\dots\omega(x_n):$ with $x_1, \dots, x_n \in \mathbb{R}^d$, i.e. polynomials $\omega(x_1)\dots\omega(x_n)$ orthogonalized.

Other representations

$\mathcal{K} = \oplus_{n=0}^{\infty} \mathcal{K}_n$ with $\mathcal{K}_n$ being the closure of all symmetric functions $u(x_1, \dots, x_n)$ with $x_1, \dots, x_n \in \mathbb{R}^d$ endowed with the scalar product

$$(u, v) := \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dn}} \overline{u(\mathbf{x})} C_n(\mathbf{y} - \mathbf{x}) v(\mathbf{y}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y}$$

where $C_n(\mathbf{y} - \mathbf{x}) = \prod_{m=1}^{n} C(y_m - x_m)$.

Gaussian embedding:
$u \mapsto \frac{1}{\sqrt{n!}} \int_{\mathbb{R}^{dn}} u(\mathbf{x}) : \omega(x_1) \dots \omega(x_n) : \mathrm{d}\mathbf{x}$.

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Fourier space: $\widehat{\mathcal{K}} = \oplus_{n=0}^{\infty} \widehat{\mathcal{K}}_n$ where $\widehat{\mathcal{K}}_n$ contains the symmetric functions $\widehat{u}(p_1, \dots, p_n)$ with the scalar product

$$(\widehat{u}, \widehat{v}) := \int_{\mathbb{R}^{dn}} \overline{\widehat{u}(\mathbf{p})} \widehat{C}_n(\mathbf{p}) \widehat{v}(\mathbf{p}) \, \mathrm{d}\mathbf{p}$$

where $\widehat{C}_n(\mathbf{p}) = \prod_{m=1}^{n} \widehat{C}(p_m)$.

The spaces $\mathcal{H}_n$, $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_n$ are unitary equivalent.

Operators

Differentiation in the lth direction:
$\nabla_l \widehat{u}(\mathbf{p}) = i\left(\sum_{m=1}^{n} p_{ml}\right) \widehat{u}(\mathbf{p})$

Laplacian: $\Delta = \sum_{l=1}^{d} \nabla_l^2$
$\Delta \widehat{u}(\mathbf{p}) = -\left|\sum_{m=1}^{n} p_m\right|^2 \widehat{u}(\mathbf{p})$

Creation: $a_l^* \widehat{u}(p_1, \dots, p_{n+1}) =$
$\frac{1}{\sqrt{n+1}} \sum_{m=1}^{n+1} \widehat{u}(p_1, \dots, p_{m-1}, p_{m+1}, \dots, p_{n+1}) i p_{ml}$

Annihilation:
$a_l \widehat{u}(p_1, \dots, p_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} \widehat{u}(p_1, \dots, p_{n-1}, q) i q_l \widehat{C}(q) \, \mathrm{d}q$

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$\frac{1}{\sqrt{n+1}}\sum_{m=1}^{n+1}\widehat{u}(p_1,\dots,p_{m-1},p_{m+1},\dots,p_{n+1})ip_{ml}$

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Infinitesimal generator of $\eta(t)$ – i.e. an operator G acting on $L^2(\Omega,\pi)$ defined by

$$(Gf)(\omega) = \lim_{\mathrm{d}t\to 0}\frac{\mathbf{E}\left(f(\eta(t+\mathrm{d}t)) - f(\eta(t))\;\middle|\;\eta(t)=\omega\right)}{\mathrm{d}t}$$

for all $f\in L^2(\Omega,\pi)$:

$$G = \frac{1}{2}\Delta + \sum_{l=1}^d\left(a_l^*\nabla_l + \nabla_l a_l\right)$$

Kipnis – Varadhan theory

General setup: $\eta(t)$ is a stationary and ergodic Markov process on the state space (Ω, π).

G is the infinitesimal generator of $\eta(t)$ acting on $L^2(\Omega, \pi)$.
Notation: $S := -\frac{1}{2}(G + G^*)$ and $A := \frac{1}{2}(G - G^*)$.
$\varphi \in L^2(\Omega, \pi)$ with $\int_\Omega \varphi \,\mathrm{d}\pi = 0$.

Question: sufficient condition for the martingale approximation and central limit theorem for

$$Y_N(t) := \frac{1}{\sqrt{N}} \int_0^{Nt} \varphi(\eta(s)) \,\mathrm{d}s.$$

Sufficient conditions

- C. Kipnis, S. R. S. Varadhan, 1986 (reversible)
- B. Tóth, 1986 (non-reversible, discrete time)
- S. V. S. Varadhan, 1996: *(strong) sector condition*

$$\|S^{-1/2}AS^{-1/2}\| < \infty.$$

- S. Sethuraman, S. R. S. Varadhan, H-T. Yau, 2000: *graded/weak sector condition* $L^2(\Omega,\pi) = \oplus_{n=0}^{\infty}\mathcal{H}_n$ and
  - $S = \sum_n S_n$ with $S_n : \mathcal{H}_n \to \mathcal{H}_n$ and
  - $A = \sum_n A_{n+} + A_{n-}$ with $A_\pm : \mathcal{H}_n \to \mathcal{H}_{n\pm1}$

$$\left\|S_{n+1}^{-1/2}A_{n+}S_n^{-1/2}\right\| \le Cn^\gamma \quad \text{with } \gamma < 1.$$

CLT for the SRBP

$$G = \underbrace{\frac{1}{2}\Delta}_{-S} + \underbrace{\sum_{l=1}^{d}(a_l^*\nabla_l + \nabla_l a_l)}_{A}$$

For the graded sector condition:
$S_{n+1}^{-1/2}A_{n+}S_n^{-1/2} = \sum_{l=1}^d \left|\frac{1}{2}\Delta\right|^{-1/2} a_l^*\nabla_l \left|\frac{1}{2}\Delta\right|^{-1/2}$.
Since $\Delta = \sum_{l=1}^d \nabla_l^2$,

$$\left\|\nabla_l|\Delta|^{-1/2}\right\| \leq 1.$$

Computations yield that in at least 3 dimensions:

$$\left\||\nabla|^{-1/2}a_l^* \restriction_{\mathcal{H}_n}\right\| \leq C\sqrt{n}.$$

□

The end

Thank you for your attention!