The geometry of coalescing random walks, the Brownian web distance and KPZ universality

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Outline

- Introduction
- Random walk web distance
- Brownian web and Brownian web distance
- Main results: properties of Brownian web distance
- Convergence of random walk web distance to Brownian web distance
- KPZ limit

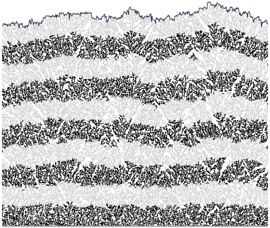
Introduction

joint work with Bálint Virág (arxiv: 2306.09073)

KPZ class models

Motivation: description of surface growth, e.g.

- boundary evolutions
- paper wetting and burning fronts
- bacterial colonies



Kardar–Parisi–Zhang (KPZ) equation, 1986:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$$

where ξ is 2D white noise

Universality and scaling

KPZ universality conjecture, 1:2:3 scaling: for a wide class of surface growth models with height function $h(t,x)$,

$$\frac{h(n^{3/3}t, n^{2/3}x) - \mathrm{E}(h(nt, n^{2/3}x))}{n^{1/3}}$$

converges as $n \to \infty$

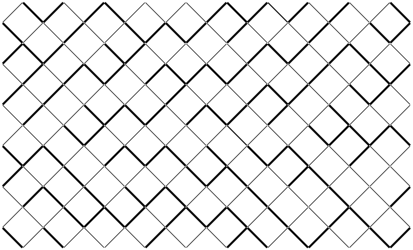
Directed landscape $\mathcal{L}(x,t;y,s)$: universal joint scaling limit of the height difference $h(ns, n^{2/3}y) - h(nt, n^{2/3}x)$ (Dauvergne, Ortmann, Virág, 2018)

Other universality classes: e.g.

- Edwards–Wilkinson: 1:2:4 scaling: additive stochastic heat equation
- Brownian castle (Hairer–Cannizzaro, 2022): 1:1:2 scaling: Brownian motion on the Brownian web

Random walk web

- Lattice:
  $\{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$
  with directed lattice edges from (i,n) to $(i+1, n\pm 1)$
- Graph of free edges: one of the outgoing lattice edges everywhere with equal probabilities independently, i.e. coalescing random walks to the right



Random walk web

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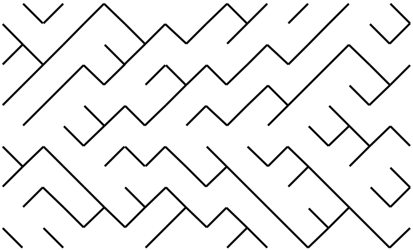
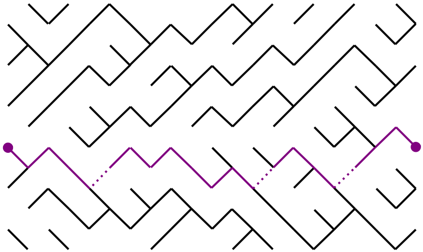
- Graph of free edges: one of the outgoing lattice edges everywhere with equal probabilities independently, i.e. coalescing random walks to the right
- Edge weights: edges of the graph with weight 0, other lattice edges with weight 1
- Distance $D^{\mathrm{RW}}(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight



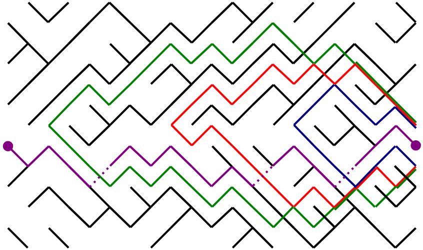
Random walk web distance

- Distance $D^{\mathrm{RW}}(i, n; j, m)$: weight of the directed path between (i, n) and (j, m) with minimal total weight
- In other words: minimal number of jumps to get from (i, n) to (j, m)



Random walk web distance

- Distance $D^{\mathrm{RW}}(i, n; j, m)$: weight of the directed path between (i, n) and (j, m) with minimal total weight
- In other words: minimal number of jumps to get from (i, n) to (j, m)
- Blue, red, green regions: set of starting points with 0, 1 and 2 jumps to the purple target point
- Aim: distance function between remote points, scaling, continuum limit



Brownian web and its dual

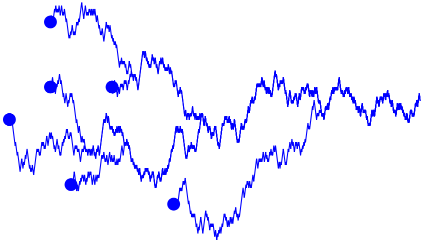
Brownian web: coalescing Brownian motions starting at all $(t, x) \in \mathbb{R}^2$

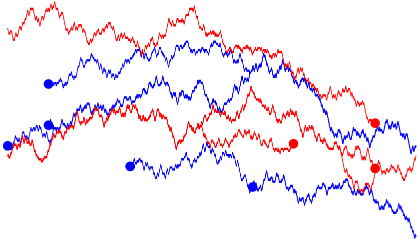
History:

- Arratia, 1979, unpublished
- Tóth, Werner, 1998, construction, special points, local time of true self-repelling motion
- Fontes, Isopi, Newman, Ravishankar, 2004, topology, „Brownian web"

Dual: coalescing backward Brownian motions

Forward and backward paths do intersect but they do not cross





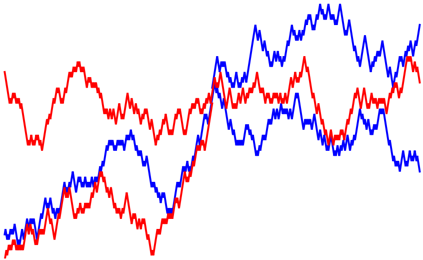
Special points of the Brownian web

Special points: point of type $(m_{\text{in}}, m_{\text{out}})$ has m_{in} incoming and m_{out} outgoing paths

Possible types: $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(2,1)$

Almost all points of $\mathbb{R}^2$ are of type $(0,1)$

Characterization of $(1,2)$ points (see figure): those hit by a forward and a backward path



Brownian web distance

Brownian web distance $D^{\mathrm{Br}}(t,x;s,y)$: minimal number of jumps to get from (t,x) to (s,y) using Brownian web paths and with jumps at $(1,2)$ points

Basic properties:

- D^{Br} is integer valued
- D^{Br} is non-symmetric
- $D^{\mathrm{Br}}(t,x;t,x)=0$
- Triangle inequality:

$$D^{\mathrm{Br}}(t,x;s,y) \le D^{\mathrm{Br}}(t,x;u,z) + D^{\mathrm{Br}}(u,z;s,y)$$

- $D^{\mathrm{Br}}(t,x;s,y)=\infty$ for a typical (s,y) which is not hit by a Brownian web path

Main results

0:1:2 scale invariance (c.f. 1:2:3 scaling in the KPZ class):

Proposition

For all $\alpha > 0$, *it holds that*

$$D^{\mathrm{Br}}(\alpha^2 t, \alpha x; \alpha^2 s, \alpha y) \stackrel{\mathrm{d}}{=} D^{\mathrm{Br}}(t, x; s, y).$$

Convergence:

Theorem (B. V., B. Virág, 2023)

- *The Brownian web distance as a function* $D^{\mathrm{Br}} : \mathbb{R}^4 \to \mathbb{R} \cup \{\infty\}$ *is almost surely lower semicontinuous.*
- *There is a coupling of the underlying random walk webs and Brownian web such that*

$$D^{\mathrm{RW}}(nt, n^{1/2}x; ns, n^{1/2}y) \to D^{\mathrm{Br}}(t, x; s, y)$$

as $n \to \infty$ *almost surely in the epigraph sense.*

KPZ limit after a shear mapping

Brownian web distance:

Theorem (B. V., B. Virág, 2023)

As $m \to \infty$, we have that

$$\frac{tm + 2zm^{2/3} - D^{\mathrm{Br}}(-tm, 2tm + 2zm^{2/3}; 0, \mathbb{R}_-)}{m^{1/3}} \to \mathcal{L}(0, 0; z, t)$$

where $\mathcal{L}$ is the directed landscape.

Random walk web distance:

Theorem (B. V., B. Virág, 2023)

For any $\eta \in (0,1)$, we have that

$$\frac{c_1(\eta)n - c_2(\eta)zn^{2/3} - D^{\mathrm{RW}}(-n, \eta n + c_3(\eta)zn^{2/3}; 0, \mathbb{Z}_-)}{c_4(\eta)n^{1/3}} \to \mathcal{L}(0, 0; z, 1)$$

as $n \to \infty$ where $\mathcal{L}(0,0;z,1) = \mathcal{A}(z) - z^2$ is the parabolic Airy process.

Horizontal scaling of random walk web distance

Theorem (B. V., B. Virág, 2023)

There is a $c > 1$ *such that*

$$\mathrm{P}\left(1/c \le \frac{D^{\mathrm{RW}}(0,0;n,0)}{\log n} \le c\right) \to 1$$

as $n \to \infty$.

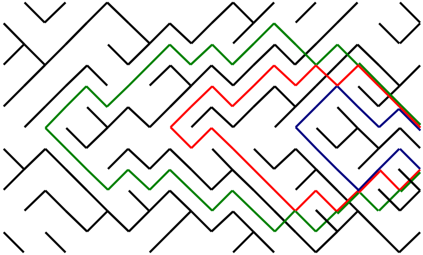
Convergence of random walk web distance: regions

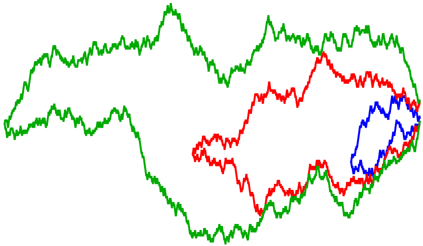
Blue, red, green regions: set of starting points with 0, 1 and 2 jumps to the target point on the right

Let $r_k^\pm$ and $\rho_k^\pm$ be the boundaries of the set of starting points with at most k jumps for D^{RW} and D^{Br}.

Evolution of r_k^+ given $r_0^+, \dots, r_{k-1}^+$: random walk reflected off r_{k-1}^+ in the discrete Skorokhod sense

Evolution of ρ_k^+ given $\rho_0^+, \dots, \rho_{k-1}^+$: Brownian motion reflected off ρ_{k-1}^+ in the Skorokhod sense





Convergence of region boundaries

Let $\widehat{Y}_{(j,m)}(i)$ for $i = j, j-1, \dots$ denote the backward random walk in the dual random walk web starting at (j, m).
Let $\widehat{B}_{(s,y)}(t)$ for $t \le s$ denote the backward Brownian motion in the dual Brownian web starting at (s, y).

When the targets are $j \times \mathbb{Z}_-$ and $s \times \mathbb{R}_-$, then $r_0^+ = \widehat{Y}_{(j,0)}$ and $\rho_0^+ = \widehat{B}_{(s,0)}$.
Inductive characterization of region boundaries:

$$r_k^+(i) = \max_{l \in \{i,\dots,j\}} \widehat{Y}_{(l, r_{k-1}(l+1)+1)}(i),$$
$$\rho_k^+(t) = \sup_{q \in [t,s]} \widehat{B}_{(q, \rho_{k-1}^+(q))}(t).$$

The random walk webs and the Brownian web can be coupled so that any backward random walk path $\widehat{Y}_{(l, r_{k-1}(l+1)+1)}$ converge almost surely to a backward Brownian path starting at some $(q, \rho_{k-1}^+(q))$.
Hence almost surely $\limsup_{n\to\infty} n^{-1/2} r_k^+(nq) \le \rho_k^+(q)$.
But $\lim_{n\to\infty} n^{-1/2} r_k^+(nq) = \rho_k^+(q)$ in law.

KPZ limit of Brownian web distance after a shear mapping

Brownian last passage percolation (BLPP):

$$L(t,n) = \sup_{0=t_{-1}\le t_0\le\cdots\le t_n=t} \sum_{i=0}^{n} (W_i(t_i) - W_i(t_{i-1}))$$

where $W_0, W_1, W_2, \dots$ are independent standard Brownian motions.

Recursion gives Skorokhod reflection:

$$L(t,n) = W_n(t) - \inf_{s\in[0,t]} (W_n(s) - L(t,n-1)).$$

If the target interval is $\{0\}\times\mathbb{R}_-$, then for the boundary

$$\rho_{tn+2zn^{2/3}}(-t) \overset{\mathrm{d}}{=} L(t, tn+2zn^{2/3}) \overset{\mathrm{d}}{=} \frac{1}{\sqrt{n}} L(tn, tn+2zn^{2/3})$$

using the Brownian scaling. The fluctuations of BLPP are known to satisfy

$$\frac{L(tn, tn+2zn^{2/3}) - 2tn - 2zn^{2/3}}{n^{1/3}} \to \mathcal{L}(0,0;z,t).$$

The end

Thank you for your attention!