Budapest University of Technology and Economics

# Directed random polymer models and the Kardar-Parisi-Zhang equation

*Author:*
Zsófia Talyigás

*Supervisor:*
Dr. Bálint Vető

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# Contents

# Introduction

## Statistical physics

A polymer is a large molecule, or macromolecule, composed of many repeated subunits. The problem of modeling polymers' formation and long time behavior became popular among physicists and mathematicians due to their crucial importance in chemistry, biology and physics. Tools of statistical physics and probability are well-suited for constructing and analyzing polymer models. This work is based on two recent articles [1] and [2] that investigate such models. A statistical physical introduction from [3] explains what a partition function is which we are interested in this work.

According to statistical mechanics, the probability that a system in thermal equilibrium occupies a state with the energy $E$ is proportional to its Boltzmann weight $e^{\frac{E}{k_B T}}$, where $T$ is the absolute temperature and $k_B$ is the Boltzmann constant. L. Boltzmann considered a gas of identical molecules which exchange energy upon colliding but otherwise are independent of each other. An individual molecule of such a gas does not have a constant velocity, so that no exact statement can be made concerning its state at a particular time. However, when the gas comes to equilibrium at some fixed temperature, one can make predictions about the average fraction of molecules which are in a given state. These average fractions are equivalent to probabilities and therefore the probability distribution for a molecule over its possible states can be introduced. Let the set of energies available to each molecule be denoted by $\{\epsilon_l\}$. The probability, $P_l$, of finding a molecule in the state

$l$ with the energy $\epsilon_l$ is

$$P_l = \frac{\exp(-\epsilon_l/(k_B T))}{\sum_m \exp(-\epsilon_m/(k_B T))}.$$

This is called the Boltzmann distribution.

J.W. Gibbs introduced the concept of an ensemble, which is defined as a set of a very large number of systems, all dynamically identical with the system under consideration. The ensemble, also called the canonical ensemble, describes a system which is not isolated but which is in thermal contact with a heat reservoir. Since the system exchanges energy with the heat reservoir, the energy of the system is not constant and can be described by a probability distribution. Gibbs proved that the Boltzmann distribution holds not only for a molecule, but also for a system in thermal equilibrium. The probability $P(E_l)$ of finding a system in a given energy $E_l$ is

$$P(E_l) = \frac{\exp(-E_l/(k_B T))}{\sum_l \exp(-E_l/(k_B T))}, \tag{0.0.1}$$

and the sum in the denominator is called partition function.

Now let us return to polymers and consider only one particle moving between two points in a random environment. Also, let us restrict ourself to a model where the particle can only move from the initial point towards the target point. In this case we talk about directed polymers. Assume that the starting point is $(0, 1)$ and the endpoint is $(\tau, N)$ ($N \in \mathbb{N}$, $\tau \in \mathbb{R}_+$) on the plane with continuous horizontal and discrete vertical coordinates. Then a directed path should be an up-right path, as the particle can move only towards the target point. Furthermore, such a path is semi-discrete, because of the setup of the coordinate system. More exactly, a path is a union of horizontal line segments and looks like a simple function on $(0, \tau)$ mapping to $\{1, 2, \ldots N\}$, jumping (almost) always $+1$ at a jumping point. See Section 2.1.1 and Figure 2.1 for more precise description of semi-discrete up-right paths.

To a semi-discrete up-right path $\pi$ one can assign an energy $E(\pi)$, and here we connect to the Boltzmann- and Gibbs measures. We would like to choose a path randomly, so we need a probability measure. In accordance with (0.0.1) we assign to $\pi$ a Boltzmann weight $e^{-\beta E(\pi)}$, and its density will be

$$\mathrm{d}\mu(\pi) = \frac{e^{-\beta E(\pi)}}{\int \mathrm{d}\pi e^{-\beta E(\pi)}}\mathrm{d}\pi = \frac{e^{-\beta E(\pi)}}{Z(\beta)}\mathrm{d}\pi,$$

where $\beta$ is the inverse temperature multiplied by the Boltzmann weight, and $Z(\beta)$ is the partition function of the directed polymer and the integration goes over the space of all possible paths. If $\beta = 0$, this is just a uniform choice. It is called the ground state, if $\beta = \infty$, and in this case all the weight is divided among the paths that minimize the energy function, so it models the principle of minimum energy.

The reason of the great interest is the relation to KPZ equation (1.2). This equation is a non-linear stochastic partial differential equation (PDE) and it is not well posed, meaning that we a priori do not know about the regularity of its solution because of the non-linear term. However, the Hopf-Cole solution formally solves the equation in the following way: Consider the solution to the stochastic heat equation (SHE) (Definition 2.7). Take its logarithm (it exists) and apply Itô's formula for that. Then it turns out that the logarithm of the solution to the SHE is formally the solution to the KPZ equation. Due to regularity problems we do not know if this solution exists for an arbitrary initial condition. However Martin Hairer's work [4] in this area was awarded the Fields medal in 2014. In our work the partition function of a continuum directed random polymer is the solution to the SHE with initial data $\mathcal{Z}_0(X)$. Furthermore, the free energy is given by the Hopf-Cole solution to the KPZ equation with initial data $\ln(\mathcal{Z}_0(X))$. So the free energy is simply the logarithm of the partition function and so we work with them and with the SHE and KPZ equations interchangeably.

As the solution of the KPZ equation is investigated in this work, it is also related to the KPZ universality class. The KPZ universality class was introduced in the context of studying the motion of growing interfaces in a 1986 paper of Kardar, Parisi and Zhang [5] which has since been cited thousands of times in both the mathematics and physics literature. The work was based on studying a continuum stochastically growing height function given in terms of a stochastic PDE which is now known as the KPZ equation. The time derivative of the height function depends on three factors: smoothing (the Laplacian), rotationally invariant, slope dependent, growth speed (the square of the gradient), noise (space-time white noise). A growth model is considered to be in the KPZ universality class if its long time behavior is similar to that of the KPZ equation itself. There are discrete mathematical models that share the three characterizing properties of the universality class and that are expected to be in this class. According to the KPZ universality conjecture these models have similar fluctuation and statistics properties. For some of them it was shown that their long time behavior is indeed similar to that of the KPZ equation, with certain initial data. Such a model is e.g. the model for interacting particle systems and simple exclusion processes [6].

The polymer model we are working with is of course also expected to be in the universality class, since in our case the polymer's free energy is the solution to the KPZ equation itself. The KPZ universality conjecture says that the scaling factor $T^{1/3}$ for the fluctuation and the limiting fluctuation statistics (in our case the Borodin-Péché distribution from Definition 2.8) should not depend on the details of the model.

## Directed polymers

Let us give an outline, that mentions different types of polymer models, and some results showing what is known in this area. The importance of this topic lies in the relation between directed random polymers and the Kardar-Parisi-Zhang (KPZ) equation and universality class. An important progress was possible thanks to the existence of models with exact solvability properties, that is models for which, exact computations are possible. (E.g. giving a Fredholm-determinant formula for the partition function as one can see it later.) Properties that might make exact calculations possible are e.g. the exactly known stationary measure, the existence of combinatorial correspondence (Robinson-Schensted-Knuth (RSK) correspondence and geometric RSK (gRSK) correspondence) or the Bethe ansatz integrability. A recent article [7] summarizes the main results in connection with directed polymers and their exact solvability. Some of them are listed here.

The first discovered exactly solvable model of directed polymers on the square lattice at finite temperature was the Log-Gamma polymer. It was introduced because of the possibility of writing down exactly its stationary measure [8]. It was later shown that the model is exactly solvable using the gRSK correspondence [9]. The Strict-Weak polymer, introduced shortly after, also enjoys these two properties [10], that is its stationary measure is known and the gRSK correspondence is applicable. The third exact solvability property, namely the Bethe ansatz integrability has been shown for the recently discovered Beta [11] and Inverse-Beta polymers [12] (and a work on the stationary measure of the Beta polymer is currently in preparation [13]). Present work is about the O'Connel-Yor semi-discrete directed polymer and about the continuum directed random polymer (CDRP). Exact solvability properties have been published also about these models. The stationary measure of the O'Connell-Yor semi-discrete directed polymer is known [14] and it is solvable using the gRSK correspondence [15]. Furthermore, the continuum directed random polymer has an exactly known stationary measure: starting from an initial condition such that the free energy of the directed polymer performs a Brownian motion, it remains so at all times [16].

Considering models with exact solvable properties, the following topic is of great interest: the exact distribution of the fluctuations of the free energy at large scale. This is in fact the question we investigate in this work. In the literature this problem is approached with the RSK/gRSK correspondence [17] and also with the Bethe ansatz integrability [11]. However our investigation relies on earlier results for slightly different models that can be validated for our case. So the already existing exact computations in this area gave the motivation and the ideas for our work.

Now let us turn to the model we are working with. Our main focus is on the large time behavior of the free energy of a continuum directed random polymer. This CDRP is the scaling limit of a semi-discrete polymer, investigated in [2]. The semi-discrete model is a mixture of the O'Connel-Yor semi-discrete and the Log-Gamma discrete directed random polymers. Our discussion relies on the results of [2] and [1].

In [1] the O'Connel-Yor model was considered without the log-gamma weights. The large time limit of the free energy was determined in this case. In [2] a Fredholm determinant formula was given for the Laplace transform of the partition function of the above mentioned mixture of polymers. Then the model was restricted to the case when there is only one level of perturbation (one column of log-gamma variables and one Brownian motion with nonzero drift). Also it was modified such that the log-gamma weight in the corner (which explodes in the limit that approaches the stationary solution) was replaced by zero. In this setting another Fredholm determinant formula was given for the Laplace transform of the continuous partition function.

Present work proves similar statements for the general semi-discrete directed polymer model with log-gamma boundary sources (it was called mixture of polymers above). We give a Fredholm determinant formula for the Laplace transform of the continuous partition function (using e.g. the formula for the semi-discrete one, given in [2]). Furthermore, our main purpose is to give the distribution of the fluctuations of the free energy at large scale. First a restriction will be made for the case with one level of boundary perturbation, just as it was in [2]. Then we extend to the general case using similar ideas.

Let us give now a short outline of our work. In Chapter 1 we make clear the mathematical concepts that are constantly used throughout, including integral operators, Fredholm determinants, the KPZ equation and the stochastic heat equation, and we recall the definition of convergence in distribution and Lebesgue's dominated convergence theorem.

We introduce the semi-discrete directed random polymer model in Chapter 2. The semi-discrete and continuous partition functions and free energies are defined here. Besides these, we introduce a special initial data for the stochastic heat equation, which has an important role later on. Our main result, Theorem 2.9 is also stated in this chapter.

In Chapter 3 we restrict ourself to the case when there is only one level of perturbations. For this case we first determine a Fredholm determinant formula for the Laplace transform of the continuous partition function in Theorem 3.4. Then Theorem 3.9, the special case of Theorem 2.9 is proved. Finally we extend this proof to the general case in Chapter 4.

# Chapter 1

# Preliminaries

In this chapter we would like to make clear the mathematical concepts we are working with throughout this work. These are the notions of pure functional analysis, probability and stochastics. Polymer models are explained in the next chapter.

## 1.1 Integral operators and Fredholm determinants

An important result of this work is that we give a Fredholm determinant formula for the Laplace transform of the continuum directed random polymer's partition function. Before introducing the notion for a Fredholm determinant, integral operators need to be defined.

**Definition 1.1.** *[18] An integral operator is a map $f \mapsto Af$ where the law of the correspondence $A$ is given by the integral*

$$Af(t) = \int_D G(t, \tau, f(\tau))d\tau,\ t \in D \tag{1.1.1}$$

*where $D$ is a given measurable set of finite Lebesgue measure in a finite dimensional space and $G(t, \tau, u)$, $t, \tau \in D$, $-\infty < u < \infty$, is a given measurable function. It is assumed that $G$ and $f$ are functions satisfying conditions that ensure the existence* *of the integral in (1.1.1). If $G(t,\tau,u)$ is a non-linear function in $u$, then we have a non-linear integral operator. If $G(t,\tau,u) = K(t,\tau)u$, then (1.1.1) takes the form*

$$Af(t) = \int\limits_D K(t,\tau)f(\tau)d\tau,\ t \in D, \tag{1.1.2}$$

*the generated operator is called a linear integral operator and the function $K$ is called its kernel.*

Now we can turn to the Fredholm determinants. This is how one can compute the determinant of an operator. In our case the operator is always a linear integral operator, so we write down the definition for this case.

**Definition 1.2.** *Fredholm-determinant [2]:*
*Fix a Hilbert space $L^2(X,\mu)$ where $X$ is a measure space and $\mu$ is a measure on $X$. Let $K$ be an integral operator acting on $f(\cdot) \in L^2(X)$ by $Kf(x) = \int_X K(x,y)f(y)\,\mathrm{d}\mu(y)$, where $K(x,y)$ is the kernel of $K$ and we will assume throughout that $K(x,y)$ is continuous in both $x$ and $y$. Assuming its convergence, the Fredholm determinant expansion of $\mathbb{1}+K$ is defined as*

$$\det(\mathbb{1}+K)_{L^2(X)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X \det\left[K(x_i,x_j)\right]_{i,j=1}^n \prod_{i=1}^n \mathrm{d}\mu(x_i).$$

Now let us give an example for an important integral kernel, and for the Fredholm determinant it defines. The Airy function $Ai(x)$ is the solution of the Airy equation $y'' = xy$. It has an integral representation on the complex plane:

$$Ai(z) := \frac{1}{2\pi i} \int\limits_{\infty e^{-\frac{\pi}{3}\mathrm{i}}}^{\infty e^{\frac{\pi}{3}\mathrm{i}}} e^{\frac{t^3}{3} - tz}\mathrm{d}t. \tag{1.1.3}$$

and the Airy kernel is given by

$$K_{Ai}(x,y) := \int\limits_0^{\infty} Ai(x+\lambda)Ai(y+\lambda)d\lambda. \tag{1.1.4}$$

The Fredholm determinant, this kernel defines, is the distribution function of the Tracy-Widom distribution. It is the limiting distribution of the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble (GUE). Let $A$ denote the integral operator given by the Airy kernel $K_{Ai}$. Then the distribution function $F_{GUE}$ of the Tracy-Widom distribution can be written as the following Fredholm determinant:

$$F_{GUE}(r) = \det\left(\mathbb{1} - A\right)_{L^2(r,\infty)}, \quad r \in \mathbb{R} \tag{1.1.5}$$

Beyond that this is an important example in general, it is also significant for us. We define the Borodin-Péché distribution later on. Its distribution function is also given by a Fredholm determinant, in fact, its kernel is a generalized Airy kernel. Furthermore, the Borodin-Péché distribution is related to random matrix theory. In Chapter 2 we mention the role of the Tracy-Widom distribution in the KPZ universality classes, which the Borodin-Péché distribution also has, being the large time limiting statistics for the free energy fluctuation.

## 1.2 Kardar-Parisi-Zhang (KPZ) equation

As we mentioned in the introduction Kardar, Parisi and Zhang proposed the stochastic evolution equation for a height function $\mathcal{F}(T,X) \in \mathbb{R}$ ($T \in \mathbb{R}_+$ is time and $X \in \mathbb{R}$ is space)

$$\partial_t \mathcal{F}(T,X) = \frac{1}{2}\partial_X^2 \mathcal{F}(T,X) + \frac{1}{2}(\partial_X \mathcal{F}(T,X))^2 + \xi(T,X), \quad \mathcal{F}(0,X) = \mathcal{F}_0(X), \tag{1.2.1}$$

where $\xi$ denotes the space-time Gaussian white noise with $\mathbb{E}\left[\xi(T,X)\xi(S,Y)\right] = \delta(T-S)\delta(X-Y)$. It can also be found in the introduction that this stochastic partial differential equation is ill-posed. However one can give a formal solution indirectly via the well-posed stochastic heat equation (SHE):

$$\partial_T \mathcal{Z}(T,X) = \tfrac{1}{2}\partial_X^2 \mathcal{Z}(T,X) + \mathcal{Z}(T,X)\xi(T,X), \quad \mathcal{Z}(0,X) = \mathcal{Z}_0(X). \tag{1.2.2}$$

Now the Hopf-Cole solution to the KPZ equation is defined as

$$\mathcal{F}(T, X) = \ln \mathcal{Z}(T, X), \quad \mathcal{F}(0, X) = \ln \mathcal{Z}(0, X), \tag{1.2.3}$$

where $\mathcal{Z}(T, X)$ is the solution to the SHE. In the continuum directed random polymer model we are going to investigate, $\mathcal{Z}(T, X)$ is the partition function and $\mathcal{F}(T, X)$ is the free energy. It is going to be explained in Chapter 2, why $\mathcal{Z}$ can be called a partition function.

## 1.3 Convergences

Convergence in distribution and the interchangeability of the integral and the limit will be essential in our discussion. For instance, in our main statement Theorem 2.9 we need to prove a convergence in distribution. Furthermore, by proving convergence of Fredholm determinants we need nothing else but upper bounds and Lebesgue's dominated convergence theorem. So let us recall the definition and the theorem.

**Definition 1.3.** *A sequence $X_1, X_2, \ldots$ of random variables is said to converge in distribution to a random variable $X$, if*

$$\lim_{n \to \infty} F_n(x) = F(x), \ \forall x \in \mathbb{R},$$

*where $F_n$ and $F$ are the cumulative distribution functions of $X_n$ and $X$ respectively.*

Convergence in distribution can be defined equivalently by terms of expectations in the following way:

**Definition 1.4.** *A sequence $X_1, X_2, \ldots$ of random variables converges in distribution to a random variable $X$ if and only if*

$$\lim_{n \to \infty} \mathbb{E}\left[f(X_n)\right] = \mathbb{E}\left[f(X)\right],$$

*for any bounded, continuous function $f$.*

**Theorem 1.5** (Lebesgue dominated convergence theorem)**.** *Let* $\{f_n\}$ *be a sequence of real-valued measurable functions on a measure space* $(S, \mathcal{A}, \mu)$*. Suppose that the sequence converges pointwise to a function* $f$ *and is dominated by some integrable function* $g$ *in the sense that* $|f_n(x)| \leq g(x)$ *for all* $n$ *in the index set and for all* $x \in S$*. Then* $f$ *is integrable and*

$$\lim_{n\to\infty} \int_S f_n \mathrm{d}\mu = \int_S f \mathrm{d}\mu$$

*, moreover,*

$$\lim_{n\to\infty} \int_S |f_n - f| \mathrm{d}\mu = 0.$$

# Chapter 2

# Directed random polymer models

## 2.1 Semi-discrete directed random polymer with boundary sources

The basic setup presented here is the same as the one published in [2]. However the main results are valid for a slightly different model. Let us see first the description of the model from [2].

### 2.1.1 Semi-discrete up-right paths

This model is a mixture of models introduced by O'Connell and Yor [14, 15] and Seppäläinen [8]. Indeed, taking $M = 0$ and $\tau > 0$ recovers the semi-discrete directed random polymer of [15] while taking $M > 0$ and $\tau = 0$ recovers the log-gamma discrete directed random polymer of [8].

For $\theta > 0$, a random variable $X$ is distributed as $\Gamma(\theta)$ (written $X \sim \Gamma(\theta)$) if it has density with respect to Lebesgue measure given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(X \leq x) = \mathbb{1}_{\{x>0\}}\frac{1}{\Gamma(\theta)}\,x^{-\theta-1}e^{-x}$$

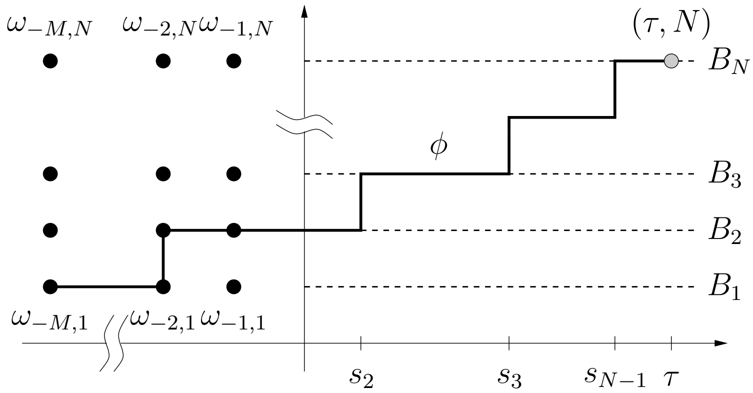


FIGURE 2.1: Illustration of the semi-discrete directed random polymer with log-gamma boundary sources. The thick solid line is a possible directed random polymer path $\phi$ from $(-M,1)$ to $(\tau,N)$. Its energy is given by (2.1.1). The random variables $\omega_{-k,n}$ are distributed as $-\ln\Gamma(\alpha_k-a_n)$, while the Brownian motions $B_1,\dots,B_N$ have drifts $a_1,\dots,a_N$ respectively.

and a random variable $W$ is distributed as $-\ln\Gamma(\theta)$ (written $W\sim-\ln\Gamma(\theta)$, called log-gamma random variable) if $W=-\ln X$ for $X\sim\Gamma(\theta)$.

Fix $N\geq 1$ and $M\geq 0$. Let $a=(a_1,\dots,a_N)\in\mathbb{R}^N$ and $\alpha=(\alpha_1,\dots,\alpha_M)\in\big(\mathbb{R}_+\big)^M$ be such that $\alpha_k-a_n>0$ for all $1\leq n\leq N$ and $1\leq k\leq M$. Consider the setting as in Figure 2.1, where the horizontal axis is discrete on the left of 0 and continuous on the right of 0, while the vertical axis is discrete. In this semi-discrete setting we introduce randomness in the following way. For all $1\leq k\leq M$ and $1\leq n\leq N$ let $\omega_{-k,n}\sim-\ln\Gamma(\alpha_k-a_n)$ be independent log-gamma random variables specified by the parameters $a,\alpha$; and for all $1\leq n\leq N$ let $B_n$ be independent Brownian motions with drift $a_n$. The $\omega_{-k,n}$ can be thought of as sitting at the lattice points $(-k,n)$ while the $B_n$ can be thought of as sitting along the horizontal rays from $(0,n)$. We denote by $\mathbb{P}$ and $\mathbb{E}$ the probability measure and expectation with respect to these random variables.

A discrete up-right path $\phi^d$ from $(i_1,j_1)$ to $(i_\ell,j_\ell)$ (written as $\phi^d:(i_1,j_1)\nearrow(i_\ell,j_\ell)$) is an ordered set of points $\big((i_1,j_1),(i_2,j_2),\dots,(i_\ell,j_\ell)\big)$ with each $(i_k,j_k)\in\mathbb{Z}^2$ and each increment $(i_k,j_k)-(i_{k-1},j_{k-1})$ either $(1,0)$ or $(0,1)$. A semi-discrete up-right path $\phi^{sd}$ from $(0,n)$ to $(\tau,N)$ (written as $\phi^{sd}:(0,n)\nearrow(\tau,N)$) is a union of horizontal line segments $\big((0,n)\to(s_n,n)\big)\cup\big((s_n,n+1)\to(s_{n+1},n+1)\big)\cup$

$\cdots \big((s_{N-1},N) \to (\tau,N)\big)$ where $0 \le s_n < s_{n+1} < \cdots < s_{N-1} \le \tau$. It is convenient to think of $\phi^{sd}$ as a surjective non-decreasing function from $[0,\tau]$ onto $\{n,\ldots,N\}$.

As we are working with a mixture of a discrete and semi-discrete lattice, our up-right paths $\phi$ will be composed of discrete portions $\phi^d$ adjoined to a semi-discrete portions $\phi^{sd}$ in such a way that for some $1 \le n \le N$, $\phi^d : (-M,1) \nearrow (-1,n)$ and $\phi^{sd} : (0,n) \nearrow (\tau,N)$.

### 2.1.2 Energy and partition function

To an up-right path, described above, we associate an energy:

$$\begin{aligned} E(\phi) &= \sum_{(i,j)\in\phi^d} \omega_{i,j} + \int_0^\tau dB_{\phi^{sd}(s)}(s) \\ &= \sum_{(i,j)\in\phi^d} \omega_{i,j} + B_n(s_n) + \\ &+ \big(B_{n+1}(s_{n+1}) - B_{n+1}(s_n)\big) + \ldots + \big(B_N(\tau) - B_N(s_{N-1})\big). \end{aligned} \tag{2.1.1}$$

This energy is random, as it is a function of the $\omega_{i,j}$ and $B_k$ random variables. We associate a Boltzmann weight $e^{E(\phi)}$ to each path $\phi$. The polymer measure on $\phi$ is proportional to this weight. The normalizing constant, or polymer partition function, is written as $\mathbf{Z}^{a,\alpha}(\tau,N)$ and is equal to the integral of the Boltzmann weight over the background measure on the path space $\phi$. Here $a$ and $\alpha$ denote the drift vector and the parameters of the log-gamma random variables. Formally it can be written as in the definition below.

**Definition 2.1.** *The partition function for the semi-discrete directed random polymer with log-gamma boundary sources is given as*

$$\mathbf{Z}^{a,\alpha}(\tau,N) = \sum_{n=1}^{N} \sum_{\phi^d:(-M,1)\nearrow(-1,n)} \int_{\phi^{sd}:(0,n)\nearrow(\tau,N)} e^{E(\phi)} \mathrm{d}\phi^{sd}$$

*where $E(\phi)$ is given by (2.1.1), the dependence on $a$ and $\alpha$ is described in Section 2.1.1 and $\mathrm{d}\phi^{sd}$ represents the Lebesgue measure on the simplex*

*$0 \leq s_n < s_{n+1} < \cdots < s_{N-1} \leq \tau$ with which $\phi^{sd}$ is identified. If $n = N$, take the counting measure (because there is only one possible $\phi^{sd}$ path).*

The other important quantity beside $\mathbf{Z}^{a,\alpha}(\tau, N)$ is its logarithm which is called the free energy:

**Definition 2.2.**

$$\mathbf{F}^{a,\alpha}(\tau, N) = \ln\left(\mathbf{Z}^{a,\alpha}(\tau, N)\right) \tag{2.1.2}$$

In order to see new results the following Fredholm determinant formula, provided in [2], is a key. The condition $N \geq 9$ was a technical detail needed in the proof of this formula. However, this condition has no meaningful role, as $N$ is sent to infinity in our further discussion .

**Theorem 2.3.** *[2, Thm. 2.1] Fix $N \geq 9$, $M \geq 0$ and $\tau > 0$. Let $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and $\alpha = (\alpha_1, \ldots, \alpha_M) \in \left(\mathbb{R}_+\right)^M$ be such that $\alpha_k - a_n > 0$ for all $1 \leq n \leq N$ and $1 \leq k \leq M$. For $1 \leq k \leq M$ and $1 \leq n \leq N$ let $\omega_{-k,n} \sim -\ln\Gamma(\alpha_k - a_n)$ be independent log-gamma random variables and for all $1 \leq n \leq N$ let $B_n$ be independent Brownian motions with drift $a_n$. Then for all $u \in \mathbb{C}$ with positive real part*

$$\mathbb{E}\left[e^{-u\mathbf{Z}^{a,\alpha}(\tau,N)}\right] = \det\left(\mathbb{1} + \mathrm{K}_u\right)_{L^2(\mathcal{C}_{a;\alpha;\varphi})}$$

*where the operator $\mathrm{K}_u$ is defined in terms of its integral kernel*

$$\mathrm{K}_u(v, v') =$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{D}_v} \mathrm{d}s\, \Gamma(-s)\Gamma(1+s) \prod_{n=1}^{N} \frac{\Gamma(v - a_n)}{\Gamma(s + v - a_n)} \prod_{k=1}^{M} \frac{\Gamma(\alpha_k - v - s)}{\Gamma(\alpha_k - v)} \frac{u^s e^{v\tau s + \tau s^2/2}}{v + s - v'}.$$

*The contour $\mathcal{C}_{a;\alpha;\varphi}$ is given in Definition 2.4 with any $\varphi \in (0, \pi/4)$, as is the contour $\mathcal{D}_v$. The meaning of $\det\left(\mathbb{1} + \mathrm{K}_u\right)_{L^2(\mathcal{C}_{a;\alpha;\varphi})}$ is explained in Defnition 1.2.*

The contours in Theorem 2.3 are defined such that they do not intersect the singularities and that the decay is fast enough for the integral to be convergent.

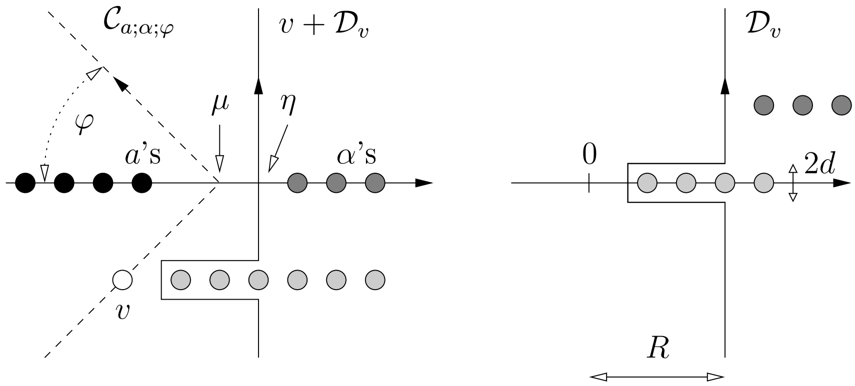


FIGURE 2.2: (Left) The contour $\mathcal{C}_{\eta;\varphi}$ (dashed) where the black dots symbolize the set of singularities of $K_u(v,v')$ in $v$ at $\cup_{1\leq n\leq N}\{a_n, a_n-1,\dots\}$ coming from the factors $\Gamma(v-a_n)$. The contour $v+\mathcal{D}_v$ is the solid line. (Right) The contour $\mathcal{D}_v$ where the light gray dots are the singularities at $\{1,2,\dots\}$ and the dark gray dots are those at $\cup_{1\leq m\leq M}\{\alpha_m-v,\alpha_m+1-v,\dots\}$ coming from $\Gamma(\alpha_m-v-s)$.

**Definition 2.4.** *Let* $a=(a_1,\dots,a_N)\in\mathbb{R}^N$ *and* $\alpha=(\alpha_1,\dots,\alpha_M)\in\left(\mathbb{R}_{>0}\right)^M$ *be such that* $\alpha_m-a_n>0$ *for all* $1\leq n\leq N$ *and* $1\leq m\leq M$. *Set* $\mu=\frac{1}{2}\max(a)+\frac{1}{2}\min(\alpha)$ *and* $\eta=\frac{1}{4}\max(a)+\frac{3}{4}\min(\alpha)$. *Then, for all* $\varphi\in(0,\pi/4)$, *we define the contour* $\mathcal{C}_{a;\alpha;\varphi}=\{\mu+e^{\mathrm{i}(\pi+\varphi)}y\}_{y\in\mathbb{R}_+}\cup\{\mu+e^{\mathrm{i}(\pi-\varphi)}y\}_{y\in\mathbb{R}_+}$. *The contours are oriented so as to have increasing imaginary part. For every* $v\in\mathcal{C}_{a;\alpha;\varphi}$ *we choose* $R=-\mathrm{Re}(v)+\eta$, $d>0$, *and define a contour* $\mathcal{D}_v$ *as follows:* $\mathcal{D}_v$ *goes by straight lines from* $R-\mathrm{i}\infty$, *to* $R-\mathrm{i}d$, *to* $1/2-\mathrm{i}d$, *to* $1/2+\mathrm{i}d$, *to* $R+\mathrm{i}d$, *to* $R+\mathrm{i}\infty$. *The parameter* $d$ *is taken small enough so that* $v+\mathcal{D}_v$ *does not intersect* $\mathcal{C}_{a;\alpha;\varphi}$. *See Figure 2.2 for an illustration.*

## 2.2 Continuum directed random polymer (CDRP)

The scaling limit of a semi-discrete partition function (or free energy) is the solution to the SHE, see (1.2.2) (or to the KPZ equation, (1.2)) with a particular initial data. This is the statement of Theorem 3.6 and more generally Theorem 4.2. The definitions of the partition function and free energy for the CDRP are defined based on this fact.

**Definition 2.5.** *The partition function* $\mathcal{Z}(T,X)$ *for the continuum directed random polymer with boundary perturbation* $\ln\mathcal{Z}_0(X)$ *is given by the solution to the* *stochastic heat equation (SHE, (1.2.2)) with multiplicative Gaussian space-time white noise and $\mathcal{Z}_0(X)$ initial data. The initial data $\mathcal{Z}_0(X)$ may be random but is assumed to be independent of the space-time white noise.*

Now we explain why $\mathcal{Z}(T,X)$ is indeed a partition function, based on [1]. This can be seen by looking at the Feynman-Kac representation of $\mathcal{Z}(T,X)$ [20]:

$$\mathcal{Z}(T,X) = \mathbb{E}_{B(X)}\left[\mathcal{Z}_0(B(0)) : \exp : \left\{\int_0^T \xi(t,B(t))dt\right\}\right], \tag{2.2.1}$$

where the expectation $\mathbb{E}$ is taken over the law of a Brownian motion $B$ which is running backwards from time $T$ and position $X$. The : exp : is the Wick exponential, see the definition e.g. in [20]. Note that the randomness of the space-time white noise remains in this formula. By time reversal we may consider this expectation as the partition function for Brownian bridges which can depart at time 0 from any location $B(0) \in \mathbb{R}$ and must end at $X$ at time $T$, picking up the weights of the space-time white noise $\xi$ on the path. Here the Wick exponential is the weight of a path, and if we want to choose a path randomly, the normalizing constant should be the integral of the weights over the space of all possible paths. This is exactly what we have on the right-hand side (RHS) of (2.2.1), and this is how one can see that this should be the scaling limit of the semi-discrete partition function.

As long as $\mathcal{Z}_0$ is almost surely positive, it follows from work of Müller [21] that, almost surely, $\mathcal{Z}(T,X)$ is positive for all $T>0$ and $X \in \mathbb{R}$. Hence we can take its logarithm.

**Definition 2.6.** *For an almost surely positive $\mathcal{Z}_0$ define the free energy for the continuum directed random polymer with initial condition $\ln \mathcal{Z}_0(X)$ as*

$$\mathcal{F}(T,X) = \ln\left(\mathcal{Z}(T,X)\right), \tag{2.2.2}$$

*that is as the Hopf-Cole solution of the KPZ equation (1.2).*

Let us present now in Definition 2.7 a particular initial data which is going to have a role later on. In fact this is the initial data that is needed to give the scaling limit of a typical semi-discrete random polymer with log-gamma boundary sources. However, this is not the most natural initial data. I. Corwin summarizes the most fundamental initial data in [22] and the knowledge in connection with their fluctuations. The initial data are given in terms of the well-posed SHE in that work. Here we list the most essential ones.

The initial data $\mathcal{Z}(0,X)$ to the SHE is called the wedge initial data. The fluctuation of the solution to the corresponding KPZ equation is distributed according to $F_{GUE}$ on the large scale, that is according to the cumulative distribution function of the Tracy-Widom random matrix distribution for the Gaussian Unitary Ensemble [23]. The $\mathcal{Z}(0,X) = 1$ is called the flat initial data, and the fluctuations' distribution at large scale is given by $F_{GOE}$, the cumulative distribution function of the Tracy-Widom GOE (Gaussian Orthogonal Ensemble) distribution [24]. Finally $\mathcal{Z}(0,X) = e^{B(X)}$ is called the stationary initial data. The limiting distribution for the fluctuation is also known in this case [25]. Now let us return to the initial data that we are working with.

**Definition 2.7.** *Fix $m \geq 1$ and $M \geq 0$. Let $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $\beta = (\beta_1, \dots, \beta_M) \in \left(\mathbb{R}_+\right)^M$ be such that $b_n < \beta_k$ for all $1 \leq n \leq m$ and $1 \leq k \leq M$. Let $B_{b,1}, B_{b,2}, \dots, B_{b,m}$ be independent Brownian motions with drifts $b_1, b_2, \dots, b_m$, and let $B_{\beta,1}, B_{\beta,2}, \dots, B_{\beta,M}$ be independent Brownian motions with drifts $\beta_1, \beta_2, \dots, \beta_M$. Furthermore, let $\omega_{-i,j} \sim -\ln\Gamma(\beta_i - b_j)$ be independent log-gamma variables. Let us create now the random variables $\mathcal{Z}^{b,\beta}(X,m)$ and $\tilde{\mathbf{Z}}^{\beta,b}(-X,M)$ for all $X \geq 0$ jointly, using the above defined Brownian motions and log-gamma variables:*

*Let $\mathcal{Z}^{b,\beta}(X,m)$ be the partition function from Definition 2.1, using $B_{b,1}, B_{b,2}, \dots, B_{b,m}$ and $\omega_{-i,j}$ $(i = 1, \dots, M,\ j = 1, \dots, m)$. Let $\tilde{\mathbf{Z}}^{\beta,b}(-X,M)$ be also a semi-discrete partition function but with the following modification: in the semi-discrete polymer with log-gamma boundary sources, the log-gamma weights are $\omega_{-i,j} \sim -\ln\Gamma(s_j - t_i)$ instead of $-\ln\Gamma(t_i - s_j)$. Thus by creating $\tilde{\mathbf{Z}}^{\beta,b}(-X,M)$ we use the Brownian motions $B_{\beta,1}, B_{\beta,2}, \dots, B_{\beta,M}$ and the log-gamma variables*

$\omega_{-i,j}$ *($i = 1, \dots, M$, $j = 1, \dots, m$). Now*

$$\mathcal{Z}_0^{b,\beta}(X) = \begin{cases} \mathbf{Z}^{b,\beta}(X, m), & \text{if } X > 0 \\ \tilde{\mathbf{Z}}^{\beta,b}(-X, M), & \text{if } X \leq 0, \end{cases} \tag{2.2.3}$$

Note that taking the modification for $\tilde{\mathbf{Z}}^{\beta,b}$ into account, we have indeed the same log-gamma variables for positive and negative $X$-s. In both cases the parameters of the gamma distributions are in the form of $\beta_i - b_j$, only the log-gamma columns become log-gamma rows (and the rows become columns) in the modified setup.

Let us introduce here the notations $\mathcal{Z}^{b,\beta}(T, X)$ and $\mathcal{F}^{b,\beta}(T, X)$. They will denote the partition function and free energy of the CDRP corresponding to the initial data defined above in Definition 2.7.

## 2.3 Large time limit – the main result

The main theorem of this work gives the limiting distribution of the free energy of the CDRP as time goes to infinity. The distribution function is given by a Fredholm determinant formula whose kernel was given by Borodin and Péché in [26]. It is referred to as Borodin-Péché distribution throughout.

**Definition 2.8.** *Fix $m \geq 1$ and $M \geq 0$. Let $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $\beta = (\beta_1, \dots, \beta_M) \in \left(\mathbb{R}_+\right)^M$, and assume that $b_1 \leq b_2 \leq \dots \leq b_m < \beta_1 \leq \beta_2 \leq \dots \leq \beta_M$. The Borodin-Péché distribution is defined as*

$$F_{BP,b,\beta}(r) = \det\left(\mathbb{1} - \mathrm{K}_{BP,b,\beta}\right)_{L^2(r,\infty)},$$

*where*

$$\mathrm{K}_{BP,b,\beta}(x, y) = \frac{1}{(2\pi\mathrm{i})^2} \int_\gamma \mathrm{d}w \int_\Gamma \mathrm{d}z \frac{1}{z - w} \frac{e^{z^3/3 - zy}}{e^{w^3/3 - wx}} \prod_{k=1}^{M} \frac{w - \beta_k}{z - \beta_k} \prod_{n=1}^{m} \frac{z - b_n}{w - b_n}. \tag{2.3.1}$$

*Let $c > 0$ be arbitrary. Then $\gamma$, the integration contour for $w$, goes from $-c - \mathrm{i}\infty$ to $-c + \mathrm{i}\infty$ such that it crosses the real axis between $b_m$ and $\beta_M$. The other contour* *for $z$, $\Gamma$ goes from $c - \mathrm{i}\infty$ to $c + \mathrm{i}\infty$ such that it also crosses the real axis between $b_m$ and $\beta_M$ and it does not intersect $\gamma$.*

In [26] this distribution was introduced as a modification of the Airy kernel with two sets of parameters. This is a generalization of the Airy kernel and also of the extended (time-dependent) version of that. It was obtained as a limit of a directed percolation in a quadrant which has both defective rows and columns. The paper also predicts that the extended kernel should appear as a scaling limit also in random matrix theory, however they could not derive it yet.

Let us state now Theorem 2.9, the main result of this work. It gives the large time limit of the CDRP free energy with the initial data defined before.

**Theorem 2.9.** *Let $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $\beta = (\beta_1, \dots \beta_M) \in \mathbb{R}_+^M$ be real vectors such that $b_j < \beta_i$ for all $1 \le j \le m$ and $1 \le i \le M$. Consider the free energy of the CDRP from Definition 2.6 with boundary perturbation $\ln \mathcal{Z}_0^{b,\beta}$, where $\mathcal{Z}_0^{b,\beta}$ is defined in Definition 2.7; and with drift vectors $\sigma b$ and $\sigma\beta$, where $\sigma = (2/T)^{1/3}$. Then for any $r \in R$,*

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{\mathcal{F}^{\sigma b, \sigma\beta}(T,0) + T/24}{(T/2)^{1/3}} \le r\right) = F_{BP,b,\beta}(r), \tag{2.3.2}$$

*where $F_{BP,b,\beta}$ is the cumulative distribution function of the Borodin-Péché distribution (see Definition 2.8).*

So we took the solution to the KPZ equation with a particular initial data, and the theorem claims that its fluctuation has Borodin-Péché distribution at large scale. As mentioned above, this distribution appeared as the limit of a percolation model which is in the KPZ universality class. Hence our theorem is in accordance with the universality conjecture.

In the next chapter the simplest $m = M = 1$ case is investigated. After understanding this instance, we will prove Theorem 2.9.

# Chapter 3

# Special case with one level of boundary perturbations

In a usual semi-discrete directed random polymer model, there are $M$ columns of log-gamma random variables and $N$ independent Brownian motions with (possibly) different drifts. Now we restrict ourselves to the case when there is only one column of log-gamma variables and every drift is zero except the first one.

A similar model was investigated in [2]. The only difference between that model and ours is the following: The former setup replaces the weight in the corner (in $(-1, 1)$) by zero, whereas we have a log-gamma weight there. Nevertheless we strongly rely on that paper and use the definitions and main theorems to find the results valid for our model.

There are two main purposes of this chapter. The first one is to give a Fredholm-determinant formula for the Laplace transform of a particular CDRP partition function in Theorem 3.4. The second aim is to give the large time limit of the free energy of the same CDRP in Theorem 3.9. This is our main result and main proof. Later, the proof of the more general statement, Theorem 2.9 will be very similar. The partition function (or free energy) in question is the solution to the SHE (or KPZ) with initial data $\mathcal{Z}_0^{b,\beta}(X)$ (or $\ln \mathcal{Z}_0^{b,\beta}(X)$) given in Definition 2.7 with $m = M = 1$.

## 3.1 Initial data

So let us first determine $\mathcal{Z}_0^{b,\beta}(X)$ in case of $m = M = 1$, because it is needed throughout this chapter. If $X > 0$ we need to compute $\mathbf{Z}^{b,\beta}(X, 1)$, so a partition function of a polymer which has only one allowed up-right path. Indeed, an up-right path in this setting starts from $(-1, 1)$, because $M = 1$, and ends in $(X, 1)$, because $m = 1$, thus no upward jump is allowed. What remains is a horizontal path, $\phi$ from $(-1, 1)$ to $(X, 1)$, with one log-gamma weight $\omega_{-1,1} \sim -\ln\Gamma(\beta - b)$ and one Brownian increment, where the Brownian motion has drift $b$ (let us denote it by $B_b$). Looking at (2.1.1) it can be seen that

$$E(\phi) = \omega_{-1,1} + B_b(X),$$

since there are no "$s_k$" jumping points. Hence in Definition 2.1 the sums have only one term and the integral just with respect to the counting measure, resulting

$$\mathbf{Z}^{b,\beta}(X, m) = \mathbf{Z}^{b,\beta}(X, 1) = e^{\omega_{-1,1} + B_b(X)}. \tag{3.1.1}$$

If $X \leq 0$, then similarly, there is only one path with energy

$$E(\phi) = \omega_{-1,1} + B_\beta(X),$$

where $B_\beta$ is a Brownian motion with drift $\beta$. Again $\omega_{-1,1} \sim -\ln\Gamma(\beta - b)$, because of the modification in Definition 2.7. Therefore

$$\tilde{\mathbf{Z}}^{\beta,b}(X, M) = \tilde{\mathbf{Z}}^{\beta,b}(X, 1) = e^{\omega_{-1,1} + B_\beta(X)}. \tag{3.1.2}$$

Knowing all of these the continuous partition function and free energy can be defined with the above calculated initial condition. This partition function was investigated also in [2].

**Definition 3.1.** *Let us denote by $\mathcal{Z}_1^{b,\beta}(T, X)$ the solution to the SHE (see (1.2.2)) with initial data $\mathcal{Z}_0(X) = \exp(B(X) + \omega_{-1,1})$, where $B(X)$ is a two-sided Brownian motion with drift $\beta$ to the left of $0$ and drift $b$ to the right of $0$, with $\beta > b$,* *and* $\omega_{-1,1} \sim -\ln\Gamma(\beta - b)$ *is a log-gamma random variable.*

*Denote furthermore* $\mathcal{F}_1^{b,\beta}(T,X)$ *the free energy for the same CDRP.*

$$\mathcal{F}_1^{b,\beta}(T,X) = \ln(\mathcal{Z}_1^{b,\beta}(T,X)) \quad \textit{with} \quad \mathcal{F}_0(X) = B(X) + \omega_{1,1}$$

*On two-sided Brownian motion we understand the following:*
$B(X) = \mathbf{1}_{X\le 0}\big(B^l(X) + \beta X\big) + \mathbf{1}_{X>0}\big(B^r(X) + bX\big)$ *where* $B^l : (-\infty, 0] \to \mathbb{R}$ *is a Brownian motion without drift pinned at* $B^l(0) = 0$*, and* $B^r : [0,\infty) \to \mathbb{R}$ *is an independent Brownian motion pinned at* $B^r(0) = 0$*.*

Note that this definition is in accordance with (3.1.1) and (3.1.2). We also introduce a notation for the semi-discrete partition function in case of $m = M = 1$.

**Definition 3.2.** *Denote* $\mathbf{Z}_1^{a,\alpha}(\tau, N)$ *the semi-discrete directed random polymer partition function with the following parameters:* $M = 1$*,* $a_1 = a$*,* $a_n \equiv 0$ *for* $n > 1$ *and* $\alpha_1 = \alpha > a$*.*

## 3.2 Fredholm determinant formula

Before stating the first important result of this chapter and giving the Fredholm determinant formula for the Laplace transform of $\mathcal{Z}_1^{b,\beta}$, we need the kernel which defines this Fredholm determinant. The definition is general for
$b_1 \le b_2 \le \cdots \le b_m < \beta_1 \le \beta_2 \le \cdots \le \beta_M$, however the claims of this chapter are stated for $m = M = 1$ (and $b_1 = b$, $\beta_1 = \beta$).

**Definition 3.3.** *Let* $b = b_1, b_2, \ldots, b_m)$ *and* $\beta = (\beta_1, \beta_2, \ldots, \beta_M)$*. Denote* $\mathrm{K}_{b,\beta}^{(\sigma)}$ *the integral operator whose kernel is*

$$\mathrm{K}_{b,\beta}^{(\sigma)}(x,y) = \frac{1}{(2\pi \mathrm{i})^2}\int \mathrm{d}w \int \mathrm{d}z \frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3 - zy}}{e^{w^3/3-wx}}\prod_{k=1}^{M}\frac{\Gamma(\sigma z - \beta_k)}{\Gamma(\sigma w - \beta_k)}\prod_{n=1}^{m}\frac{\Gamma(\sigma w - b_n)}{\Gamma(\sigma z - b_n)},$$

*where*

$$\sigma = (2/T)^{1/3}, \tag{3.2.1}$$

*and the integration contour for $w$ is from $-\frac{1}{4\sigma} - \mathrm{i}\infty$ to $-\frac{1}{4\sigma} + \mathrm{i}\infty$ and crosses the real axis between $\frac{b_m}{\sigma}$ and $\frac{\beta_1}{\sigma}$. The other contour for $z$ goes from $\frac{1}{4\sigma} - \mathrm{i}\infty$ to $\frac{1}{4\sigma} + \mathrm{i}\infty$, it also crosses the real axis between $\frac{b_m}{\sigma}$ and $\frac{\beta_1}{\sigma}$ and it does not intersect the contour for $w$.*

And now we provide the formula this section intends to justify.

**Theorem 3.4.** *Fix $S$ with positive real part, $T > 0$, $b < \beta$ real numbers and assume that $X = 0$. Set $\sigma$ as in (3.2.1). Then*

$$\mathbb{E}\left[\exp\left(-Se^{\frac{T}{24}}\mathcal{Z}_1^{b,\beta}(T,0)\right)\right] = \det\left(\mathbb{1} - \mathrm{K}_{b,\beta}^{(\sigma)}\right)_{L^2(\mathbb{R}_+)}, \tag{3.2.2}$$

*where $\mathcal{Z}_1^{b,\beta}$ is the partition function for the CDRP (Definition 3.1) and $\mathrm{K}_{b,\beta}^{(\sigma)}$ is defined above in Definition 3.3.*

There are two main theorems that lead to the desired Fredholm-determinant formula. The first one (Theorem 3.6) is the convergence of the semi-discrete partition functions to the CDRP partition function. The second one (Theorem 3.8) is the convergence of the Fredholm determinants describing the Laplace transform of the semi-discrete partition function (from Theorem 2.3). The proof of Theorem 3.4 will be quick as soon as we go through the mentioned results. However, some preparation is needed before stating and applying them.

### 3.2.1 Convergence of the semi-discrete partition function

**Definition 3.5.** *Let $\Psi(z) = \frac{d}{dz}\ln\Gamma(z)$ be the digamma function. For a given $\theta \in \mathbb{R}_+$, define*

$$\kappa(\theta) := \Psi'(\theta), \quad f(\theta) := \theta\Psi'(\theta) - \Psi(\theta), \quad c(\theta) := (-\Psi''(\theta)/2)^{1/3}.$$

*We may alternatively parameterize $\theta \in \mathbb{R}_+$ in terms of $\kappa \in \mathbb{R}_+$ as*

$$\theta_\kappa := (\Psi')^{-1}(\kappa) \in \mathbb{R}_+, \quad f_\kappa := \inf_{t>0}(\kappa t - \Psi(t)) \equiv f(\theta_\kappa), \quad c_\kappa := c(\theta_\kappa).$$

The theorem below gives the scaling limit of the semi-discrete directed random polymer's partition function $\mathbf{Z}_1^{a,\alpha}(\tau, N)$ from Definition 3.2. The scaling below is the same as the one for the O'Connel-Yor model, given in [19]:
Fix $T > 0$, $X \in \mathbb{R}$ and real numbers $b < \beta$. $\tau$ grows as $\sqrt{N}$ and the scaling factor $C$ is an exponential function of $N$, $T$ and $X$:

$$\tau = \sqrt{TN} + X \tag{3.2.3}$$

$$C(N, T, X) = \exp\left(N + \frac{1}{2}(N-1)\ln(T/N) + \frac{1}{2}\left(\sqrt{TN} + X\right) + X\sqrt{N/T}\right). \tag{3.2.4}$$

Not only the variables of the partition function but also the parameters of the polymer model are scaled, in the following way:

$$a = \vartheta + b, \quad \alpha = \vartheta + \beta, \tag{3.2.5}$$

where $\vartheta = \theta_{\sqrt{T/N}} \simeq \sqrt{N/T} + \frac{1}{2}$, with Definition 3.5.

This scaling is used in the theorem and in the corollary below. Theorem 3.6, the first important result we will employ, claims that the scaled semi-discrete partition function converges to that of the CDRP with a particular initial data given in Definition 3.1.

**Theorem 3.6.** *[19] Fix $T > 0$, $X \in \mathbb{R}$ and real numbers $b < \beta$. Consider the semi-discrete directed random polymer in Definition 3.2 with partition function $\mathbf{Z}_1^{a,\alpha}(\tau, N)$. Let the $a$ and $\alpha$ parameters of the polymer be defined as in (3.2.5). The scaling factor $C(N, T, X)$ is given by (3.2.4). Then, as $N$ goes to infinity,*

$$\frac{\mathbf{Z}_1^{a,\alpha}(\sqrt{TN} + X, N)}{C(N, T, X)} \Rightarrow \mathcal{Z}_1^{b,\beta}(T, X).$$

*The convergence is in distribution and $\mathcal{Z}_1^{b,\beta}(T, X)$ is the solution to the SHE with initial data $\exp(B(X) + \omega_{-1,1})$, see Definition 3.1.*

The proof of Theorem 3.4 is basically that both sides of (3.2.2) are the limit of $\mathbb{E}\left[e^{u\mathbf{Z}_1^{a,\alpha}(\tau,N)}\right]$ for some $u$. So let us rewrite Theorem 3.6 so that the limit is the exponent in (3.2.2). To do this let

$$u = Se^{-N-\frac{1}{2}(N-1)\ln\frac{T}{N}-\frac{1}{2}\sqrt{TN}-X\sqrt{\frac{N}{T}}+\frac{T}{24}-\frac{X}{2}+\frac{X^2}{2T}}, \tag{3.2.6}$$

where $S \in \mathbb{C}$ with positive real part. By comparing the exponents of $C(N,T,X)$ and $u$ and by Theorem 3.6 it can be seen that

$$u\mathbf{Z}_1^{a,\alpha}(\sqrt{TN}+X,N) \Rightarrow Se^{\frac{X^2}{2T}+\frac{T}{24}}\mathcal{Z}_1^{b,\beta}(T,X), \text{ as } N \to \infty \tag{3.2.7}$$

holds with the scaling applied in the theorem. The following corollary shows that this modification was useful, that is the left-hand side (LHS) of (3.2.2) can be written as the limit of the Laplace transform of the semi-discrete partition function.

**Corollary 3.7.** *Fix $T > 0$, $X \in \mathbb{R}$ and real numbers $b < \beta$. Let $\mathbf{Z}_1^{a,\alpha}(\tau,N)$ and $\mathcal{Z}_1^{b,\beta}(T,X)$ be the partition functions defined in Definition 3.2 and in Definition 3.1 respectively, and with parameters given by (3.2.5). Then for any $S$ with positive real part*

$$\mathbb{E}\left[e^{-u\mathbf{Z}_1^{a,\alpha}(\tau,N)}\right] \to \mathbb{E}\left[\exp\left(-Se^{\frac{X^2}{2T}+\frac{T}{24}}\mathcal{Z}_1^{b,\beta}(T,X)\right)\right], \text{ as } N \to \infty \tag{3.2.8}$$

*where $\tau = \sqrt{TN}+X$.*

*Proof.* By (3.2.7) the exponent on the LHS converges in distribution to the exponent on the RHS. Our statement is true due to the equivalent definition of convergence in distribution, Definition 1.4. Indeed, we took a bounded, continuous function of $u\mathbf{Z}_1^{a,\alpha}(\tau,N)$:

$$\mathbf{Z}_1^{a,\alpha}(\tau,N) > 0,$$

since it is an integral of an exponential function, and

$$\operatorname{Re} u > 0,$$

because of (3.2.6) and $\operatorname{Re} S > 0$. Thus $e^{-u\mathbf{Z}_1^{a,\alpha}(\tau,N)}$ is bounded by 1 and the expectation on the LHS must converge. $\square$

### 3.2.2 Convergence of Fredholm determinants

With Corollary 3.7 we have seen that the LHS of (3.2.2) in Theorem 3.4 is a the limit of the Laplace transform of $\mathbf{Z}_1^{a,\alpha}(\tau, N)$ as $N$ goes to infinity. We will see that the same fact can be said about the RHS of (3.2.2).

Recall the Fredholm-determinant formula (2.3) for the Laplace-transform of the semi-discrete partition function $\mathbf{Z}_1^{a,\alpha}(\tau, N)$. Giving the limit of this formula as $N$ goes to infinity, yields the Fredholm-determinant formula for the continuous partition function as well.

**Theorem 3.8.** *[2, Thm. 6.3.] Fix $S$ with positive real part, $T > 0$, $b < \beta$ real numbers and assume that $X = 0$. Set $\tau$, $a$, $\alpha$ and $\sigma$ as in (3.2.3), (3.2.5) and in (3.2.1) respectively. Use $u$ given in (3.2.6). Denote $K_u$ the integral operator defined in Theorem 2.3 and* $\mathrm{K}_{b,\beta}^{(\sigma)}$ *is given in Definition 3.3. Then*

$$\lim_{N\to\infty} \det(\mathbb{1} + \mathrm{K}_u)_{L^2(\mathcal{C}_{a_+;\alpha;\pi/4})} = \det(\mathbb{1} - \mathrm{K}_{b,\beta}^{(\sigma)})_{L^2(\mathbb{R}_+)} \tag{3.2.9}$$

*where* $a_+ = \max\{a, 0\}$.

Now we have everything to give a straightforward proof for Theorem 3.4. As mentioned before, we show that both sides of (3.2.2) are the limit of the same expectation.

*Proof of Theorem 3.4.* Fix $S$ with positive real part, $T > 0$, $b < \beta$ real numbers and assume that $X = 0$. Set $\tau$, $a$, $\alpha$ and $\sigma$ as in (3.2.3), (3.2.5) and in (3.2.1) respectively. Use $u$ given in (3.2.6). Thus the conditions of Theorem 3.8 hold, just like those of Theorem 2.3 with $\varphi = \pi/4$. Then on the one hand,

$$\lim_{N\to\infty} \mathbb{E}\left[e^{-u\mathbf{Z}_1^{a,\alpha}(\tau,N)}\right] = \lim_{N\to\infty} \det\left(\mathbb{1} - \mathrm{K}_u\right)_{L^2(\mathcal{C}_{a;\alpha;\pi/4})} = \det\left(\mathbb{1} - \mathrm{K}_{b,\beta}^{(\sigma)}\right)_{L^2(\mathbb{R}_+)}$$

by Theorem 2.3, by Theorem 3.8, and because $a_+ = a$ for large $N$. On the other hand, we know from Corollary 3.7 that

$$\lim_{N\to\infty} \mathbb{E}\left[e^{-u\mathbf{Z}_1^{a,\alpha}(\tau,N)}\right] = \mathbb{E}\left[\exp\left(-Se^{\frac{X^2}{2T}+\frac{T}{24}}\mathcal{Z}_1^{b,\beta}(T,X)\right)\right].$$

The two limits must be the same hence the theorem is proved. □

## 3.3 Large time limit

**Theorem 3.9.** *Let $b$ and $\beta$ be real numbers. Consider the free energy of the CDRP (Definition 3.1) with drift vectors $\sigma b$ and $\sigma\beta$, where $b < \beta$ and $\sigma = (2/T)^{1/3}$. Then for any $r \in R$,*

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{\mathcal{F}_1^{\sigma b,\sigma\beta}(T,0) + T/24}{(T/2)^{1/3}} \leq r\right) = F_{BP,b,\beta}(r), \tag{3.3.1}$$

*where $F_{BP,b,\beta}$ is the cumulative distribution function of the BP distribution (Definition 2.8).*

### 3.3.1 Preparation and the proof

In the course of the proof we would like to show the convergence of the Fredholm determinants by using Lebesgue's dominated convergence theorem. We are allowed to apply this theorem if there is an integrable upper bound for the absolute value of the integrand. For this aim the following results are sufficient:

- An upper bound for $\left|\mathrm{K}_{b,\beta}^{(\sigma)}\right|$:

  **Lemma 3.10.** *[2, Lemma B.4] Fix $b < \beta$ so that $\beta - b < 1$. There is a finite constant $C$ such that for any $x, y \in \mathbb{R}_+$*

  $$|\mathrm{K}_{b,\beta}^{(\sigma)}(x,y)| \leq C\exp\left(-\frac{\beta}{\sigma}y + \frac{b}{\sigma}x\right), \tag{3.3.2}$$

*see* $\mathrm{K}^{(\sigma)}_{b,\beta}$ *in Definition 3.3 and* $\sigma$ *is given by* (3.2.1).

- An upper bound for the determinant of a matrix in terms of the length of its column vectors:

  **Lemma 3.11.** *Hadamard's inequality:*
  *Let* $M$ *be the* $n \times n$ *matrix having column vectors* $v_i$*. Then*

$$|\det(M)| \leq \prod_{i=1}^{n} \|v_i\| \,. \tag{3.3.3}$$

  In particular, if the absolute value of each entry of the matrix is at most one, the upper bound is $n^{n/2}$.

Relying on these two results Proposition 3.12, i.e. the convergence of Ffredholm determinants is shown in the next section. This is the key statement that almost immediately implies Theorem 3.9.

**Proposition 3.12.**

$$\det\left(\mathbb{1} - \mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}\right)_{L^2(\mathbb{R}_+)} \to \det\left(\mathbb{1} - \mathrm{K}_{BP,b,\beta}\right)_{L^2(r,\infty)}, \ \text{as } \sigma \to 0, \tag{3.3.4}$$

*where* $\mathrm{K}^{(\sigma)}_{b,\beta}$ *and* $\mathrm{K}_{BP,b,\beta}$ *are given in Definition 3.3 and in Definition 2.8.*

We also need a probability lemma to conclude the convergence in distribution at the end of the proof.

**Lemma 3.13.** *[1, Lemma 8.1] Consider a sequence of functions* $(f_n)_{n\geq 1}$ *mapping* $\mathbb{R} \to [0,1]$ *with the following properties:*

*(a)* $f_n(x)$ *is strictly decreasing in* $x$*,* $\forall n$

*(b)* $\lim_{x\to-\infty} f_n(x) = 1$*,* $\forall n$

*(c)* $\lim_{x\to\infty} f_n(x) = 0$*,* $\forall n$

*(d)* $f_n(x) \to \mathbb{1}_{x\leq 0}$*, as* $n \to \infty$*, uniformly on* $\mathbb{R} \setminus [-\delta,\delta]$*,* $\forall \delta > 0$

*Consider a sequence of random variables $X_n$ and a continuous probability distribution function $p(r)$. Then*

$$\lim_{n\to\infty} \mathbb{E}\left[f_n(X_n - r)\right] = p(r) \quad \forall r \in \mathbb{R} \quad \Rightarrow \quad \lim_{n\to\infty} P(X_n \le r) = p(r)$$

Putting together Proposition 3.12 and Lemma 3.13, and also choosing an appropriate sequence of functions, leads to the proof of Theorem 3.9.

*Proof of Theorem 3.9.* (Based on the proof of Corollary 1.15 in [1].)
Let $\beta > b$, $S = e^{-r/\sigma}$, and let $(\Theta_T)_{T\ge 0}$ be a sequence of functions with $\Theta_T(x) = \exp(-e^{x/\sigma})$, where $\sigma = (2/T)^{1/3}$. Now observe that

$$\Theta_T\left(\frac{\mathcal{F}_1^{\sigma b,\sigma\beta}(T,0) + T/24}{\sigma^{-1}} - r\right) = \exp\left(-Se^{\mathcal{F}_1^{\sigma b,\sigma\beta}(T,0)+T/24}\right) =$$

$$= \exp\left(-Se^{T/24}\mathcal{Z}^{b,\beta}(T,0)\right). \tag{3.3.5}$$

Note furthermore, that Theorem 3.4 and Proposition3.12 apply here. Therefore, by (3.3.5) and by Definition 2.8 we conclude that the expectation of the random variable in question (LHS of (3.3.1)) converges to the cumulative distribution function of the Borodin-Péché distribution, as $\sigma \to 0$.

$$\mathbb{E}\left[\Theta_T\left(\frac{\mathcal{F}_1^{\sigma b,\sigma\beta}(T,0) + T/24}{\sigma^{-1}} - r\right)\right] = \mathbb{E}\left[\exp(-Se^{\mathcal{F}_1^{\sigma b,\sigma\beta}(T,0)+T/24})\right] =$$

$$= \det\left(\mathbb{1} - \mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}\right)_{L^2(\mathbb{R}_+)} \to \det\left(\mathbb{1} - \mathrm{K}_{BP,b,\beta}\right)_{L^2(r,\infty)} = F_{BP,b,\beta}(r) \tag{3.3.6}$$

The conditions of Lemma 3.13 hold for $f_T(x) := \Theta_T(x) = \exp(-e^{x/\sigma})$, $X_T := \frac{\mathcal{F}_1^{\sigma b,\sigma\beta}(T,0)+T/24}{\sigma^{-1}}$ and $p(r) := F_{BP,b,\beta}(r)$:

(a) $\Theta_T : \mathbb{R} \mapsto [0,1]\ \forall T$, and $\Theta_T(x)$ is strictly decreasing in $x\ \forall T$.

(b) $\lim_{x\to-\infty} \Theta_T(x) = 1,\ \forall n$

(c) $\lim_{x\to\infty} \Theta_T(x) = 0,\ \forall n$

(d) $\Theta_T(x) \to \mathbb{1}_{x\leq 0}$, as $T \to \infty$, uniformly on $\mathbb{R} \setminus [-\delta, \delta]$, $\forall \delta > 0$

In condition (d) the convergence is uniform indeed, since on $\mathbb{R} \setminus [-\delta, \delta]$, if $x > \delta$, then

$$\Theta_T(x) = \exp(-e^{x/\sigma}) < \exp(-e^{\delta/\sigma}),$$

which can be arbitrarily small independently of $x$, and if $x < -\delta$, then

$$\Theta_T(x) = \exp(-e^{x/\sigma}) > \exp(-e^{-\delta/\sigma}),$$

where $e^{-\delta/\sigma}$ can be arbitrarily close to zero, independently of $x$. These are true for any $\delta > 0$, as $T \to \infty$ (and as $\sigma \to 0$). Since $F_{BP,b,\beta}(r)$ is continuous, and $\mathbb{E}\left[\Theta_T(X_T - r)\right] \to p(r)$ due to (3.3.6), every condition holds for Lemma 3.13. It claims that the distribution function of $X_T$ converges to $p(r) = F_{BP,b,\beta}(r)$ and this was the statement of the theorem. □

### 3.3.2 Details

*Proof of Proposition 3.12.* First the convergence of the kernels is needed, then we arrive to the statement of the proposition by Lebesgue's dominated convergence theorem.

**STEP 1**

$$\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x,y) \to \mathrm{K}_{BP,b,\beta}(x+r, y+r), \text{ as } \sigma \to 0 \tag{3.3.7}$$

Proof of STEP 1:

$$\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x,y) = \frac{1}{(2\pi\mathrm{i})^2}\int \mathrm{d}w \int \mathrm{d}z \frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zy}}{e^{w^3/3-wx}}\frac{\Gamma(\sigma(\beta-z))}{\Gamma(\sigma(z-b))}\frac{\Gamma(\sigma(w-b))}{\Gamma(\sigma(\beta-w))}$$

Convergence of the first factor of the integrand:

$$\frac{\sigma\pi S^{\sigma(z-w)}}{\sigma\pi(z-w)+o(\sigma^2)} = \frac{e^{-\frac{r(z-w)\sigma}{\sigma}}}{z-w+o(\sigma)} \to \frac{e^{-r(z-w)}}{z-w}, \text{ as } \sigma \to 0$$

Convergence of the Gamma functions:

$$\frac{\Gamma(\sigma(z-b))}{\Gamma(\sigma(w-b))} \to \frac{w-b}{z-b}$$

Thus

$$\lim_{\sigma\to 0} \mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x,y) =$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int \mathrm{d}w \int \mathrm{d}z \frac{e^{-r(z-w)}}{z-w} \frac{e^{z^3/3-zy}}{e^{w^3/3-wx}} \frac{\beta-w}{\beta-z} \frac{z-b}{w-b} = \mathrm{K}_{BP,b,\beta}(x+r,y+r),$$

by Definition 2.8, and this was our claim.

Recall the definition of a Fredholm-determinant, and consider only the $n$-dimensional integral part without the summation. Our next step is to show that this integral with the scaling limit kernel converges to the integral with the Borodin-Péché kernel.

**STEP 2**

$$\lim_{\sigma\to 0} \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} \det\left[\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x_i,x_j)\right]^n_{i,j=1} \prod_{i=1}^n \mathrm{d}x_i =$$

$$= \int_r^\infty \cdots \int_r^\infty \det\left[\mathrm{K}_{BP,b,\beta}(x_i,x_j)\right]^n_{i,j=1} \prod_{i=1}^n \mathrm{d}x_i \tag{3.3.8}$$

Proof of STEP 2:

The determinant function is continuous, therefore

$$\lim_{\sigma\to 0} \det\left[\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x_i,x_j)\right]^n_{i,j=1} = \det\left[\mathrm{K}_{BP,b,\beta}(x_i,x_j)\right]^n_{i,j=1} \tag{3.3.9}$$

holds by (3.3.7). To interchange the limit in $\sigma$ and the integral, we need to find an integrable upper bound for the determinant on the LHS.

We have an upper bound for each entry of $\left[\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x_i,x_j)\right]^n_{i,j=1}$ by Lemma 3.10. Indeed, our parameters $\sigma b$ and $\sigma\beta$ are close, $\sigma$ tending to zero, hence the lemma applies. Thus the upper bound is

$$\left|\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x,y)\right| \le C\exp\left(-\beta y + bx\right), \tag{3.3.10}$$

with the conditions of Lemma 3.10. The entrywise upper bound leads to an upper bound for the determinant by Hadamard's inequality (Lemma 3.11). Let us multiply the $i$th row in the LHS by $\frac{1}{C}\exp(-bx_i)$ $\forall i = 1, \dots, n$, and the $j$th column by $\frac{1}{C}\exp(\beta x_j)$ $\forall j = 1, \dots, n$ and call this matrix $A$, with elements $A_{ij}$, $i, j = 1, \dots, n$. Using the upper bound in (3.3.10) it follows that

$$|A_{ij}| = \frac{e^{\beta x_j - b x_i}}{C} \left| \mathrm{K}^{(\sigma)}_{\sigma b, \sigma\beta}(x_i, x_j) \right| \leq \frac{e^{\beta x_j - b x_i}}{C} C e^{b x_i - \beta x_j} = 1 \tag{3.3.11}$$

Note furthermore that the construction of $A$ gives

$$\det \left[ \mathrm{K}^{(\sigma)}_{\sigma b, \sigma\beta}(x_i, x_j) \right]_{i,j=1}^{n} = C^{2n} \exp\left( \sum_{j=1}^{n} x_j \,(b - \beta) \right) \det(A) \tag{3.3.12}$$

So Lemma 3.11 and the estimation (3.3.11) give $|\det(A)| \leq n^{n/2}$, and together with (3.3.12) this means

$$\left| \det \left[ \mathrm{K}^{(\sigma)}_{\sigma b, \sigma\beta}(x_i, x_j) \right]_{i,j=1}^{n} \right| \leq C^{2n} \exp\left( \sum_{j=1}^{n} x_j \,(b - \beta) \right) n^{n/2} \tag{3.3.13}$$

Now the RHS is integrable, because $\beta > b$, hence (3.3.8) holds and STEP 2 is done.

**STEP 3**

$$\lim_{\sigma \to \infty} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} \det \left[ \mathrm{K}^{(\sigma)}_{\sigma b, \sigma\beta}(x_i, x_j) \right]_{i,j=1}^{n} \prod_{i=1}^{n} \mathrm{d}\mu(x_i) =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{r}^{\infty} \cdots \int_{r}^{\infty} \det \left[ \mathrm{K}_{BP, b, \beta}(x_i, x_j) \right]_{i,j=1}^{n} \prod_{i=1}^{n} \mathrm{d}x_i \tag{3.3.14}$$

Proof of STEP 3:

It is enough to show that after dividing by $n!$ and integrating the upper bound, given in STEP 2 (in (3.3.13)), the result will be summable. The summability would imply that the sum and the limit can be interchanged.

$$f_n(\sigma) := \frac{1}{n!} \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} \det \left[ \mathrm{K}^{(\sigma)}_{\sigma b, \sigma\beta}(x_i, x_j) \right]_{i,j=1}^{n} \prod_{i=1}^{n} \mathrm{d}x_i \leq$$

$$\leq \frac{n^{n/2}}{n!} C^{2n} \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} \exp\left( -(\beta - b) \sum_{j=1}^{n} x_j \right) \prod_{i=1}^{n} \mathrm{d}x_i =$$

$$= \frac{n^{n/2}}{n!} C^{2n} \frac{1}{(\beta - b)^n},$$

and this is summable due to D'Alambert's ratio test:

$$\limsup_{n\to\infty} \frac{(n+1)^{\frac{n+1}{2}} C^{2n+2} n! (\beta - b)^n}{(n+1)! (\beta - b)^{n+1} n^{n/2} C^{2n-2}} =$$

$$= \limsup_{n\to\infty} \left( \frac{n+1}{n} \right)^{n/2} \frac{\sqrt{n+1}}{n+1} \frac{C^4}{\beta - b} = 0 \tag{3.3.15}$$

Hence, altogether we know that the sum $\sum_{n=1}^{\infty} f_n(\sigma)$ converges uniformly, and that $f_n(\sigma)$ tends to the RHS of (3.3.12). Therefore the sum and the limit can be interchanged and STEP 3 is done, implying Proposition 3.12 is proved. □

# Chapter 4

# General case

## 4.1 Large time limit

Let us return to the typical semi-discrete directed polymers with log-gamma boundary sources. So given $M$ columns and $N$ rows of independent log-gamma variables, $\omega_{-i,j} \sim -\ln\Gamma(\alpha_i - a_j)$ and $N$ independent Brownian motions $B_i$ with drifts $a_i$. The setting is such that $a_i < \alpha_j$ for all $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. It is also assumed now that there are $m \leq N$ Brownian motions with nonzero drifts and $(a_1, a_2, \ldots, a_N) = (a_1, a_2, \ldots, a_m, 0, \ldots, 0)$. Recall that the energy of an up-right path in this setting was given by (2.1.1). Furthermore, the partition function $\mathbf{Z}^{a,\alpha}$ and free energy $\mathbf{F}^{a,\alpha}$ of such a polymer were defined in Definition 2.1 and Definition 2.2. It was mentioned that $\mathcal{Z}^{b,\beta}$, the solution to the SHE and the partition function for the CDRP (Definition 2.5), is going to be the scaling limit of $\mathbf{Z}^{a,\alpha}$. We also gave the corresponding initial data to the SHE in Definition 2.7. This claim is stated precisely in Theorem 4.2 is used in our discussion.

The purpose of this chapter is to prove Theorem 2.9, that is to prove that the large time limit of the CDRP free energy is the Borodin-Péché distribution. It is the generalization of Theorem 3.9 and its proof is pretty similar. This is the reason for being a little bit sketchy, but, considering the previously proved statements, correct. The corresponding statements for the $m = M = 1$ case are indicated in brackets at each main step. So the proof gives an outline of the proof from Chapter 3, too.

*Proof of Theorem 2.9.* – Summary

1. $\mathbb{E}\left[\exp\left(-Se^{\frac{T}{24}}\mathcal{Z}^{b,\beta}(T,0)\right)\right] = \det\left(\mathbb{1} - \mathrm{K}_{b,\beta}^{(\sigma)}\right)_{L^2(\mathbb{R}_+)}$ (Theorem 3.4)

   Proof of Step 1.:

   (a) $\frac{\mathbf{Z}^{a,\alpha}(\sqrt{TN}+X,N)}{C(N,m,T,X)} \Rightarrow \mathcal{Z}^{b,\beta}(T,X)$ (Theorem 3.6) The precise statement can be found in Theorem 4.2 below.

   (b) $\mathbb{E}\left[e^{-u\mathbf{Z}^{a,\alpha}(\tau,N)}\right] \to \mathbb{E}\left[\exp\left(-Se^{\frac{X^2}{2T}+\frac{T}{24}}\mathcal{Z}^{b,\beta}(T,X)\right)\right]$, as $N \to \infty$

   with an appropriate $u$. (Corollary 3.7)

   Indeed, consider the statement of Corollary 3.7 with

$$u = \frac{S}{C(N,m,T,X)} \exp\left(\frac{X^2}{2T} + \frac{T}{24}\right) =$$

$$S\exp\left(-N - \frac{1}{2}(N-m)\ln\left(\frac{T}{N}\right) - \frac{1}{2}\left(\sqrt{TN}+X\right) - X\sqrt{\frac{N}{T}} + \frac{X^2}{2T} + \frac{T}{24}\right).$$

   With $S$ having positive real part and $Z^{a,\alpha}(\tau,N) > 0$ for positive $\tau$ and $N$ one can tell the same proof as for Corollary 3.7.

   (c) $\mathbb{E}\left[e^{-u\mathbf{Z}^{a,\alpha}(\tau,N)}\right] = \det\left(\mathbb{1} + \mathrm{K}_u\right)_{L^2(\mathcal{C}_{a;\alpha;\varphi})}$ (Theorem 2.3)

   This is a general statement. It was not specified for $m = M = 1$.

   (d) $\lim_{N\to\infty} \det(\mathbb{1} + \mathrm{K}_u)_{L^2(\mathcal{C}_{a_+;\alpha;\pi/4})} = \det(\mathbb{1} - \mathrm{K}_{b,\beta}^{(\sigma)})_{L^2(\mathbb{R}_+)}$, (Theorem 3.8)

   Theorem 3.8 was cited from [2], where it was stated for $\beta$ and $b$ being real numbers and not vectors. However, with minor technical modifications the proof can be performed for the general case as well. Thus one can state Theorem 3.8 also for $a$, $\alpha$, $b$ and $\beta$ vectors used in this chapter.

   Statement (b) is a corollary of (a). Then Putting together (b), (c) and (d), Step 1. is done by the same argument as the one for Theorem 3.4. Thus we gained a Fredholm determinant formula for the Laplace transform of the continuous partition function.

2. $\det\left(\mathbb{1}-\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}\right)_{L^2(\mathbb{R}_+)} \to \det\left(\mathbb{1}-\mathrm{K}_{BP,b,\beta}\right)_{L^2(r,\infty)}$, as $\sigma \to 0$ (Proposition 3.12)

   Proof of Step 2.:

   (a) $\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}(x,y) \to \mathrm{K}_{BP,b,\beta}(x+r,y+r)$, as $\sigma \to 0$ (Eq. 3.3.7)

   We have totally the same limit as in (3.3.7) with more factors.

   (b) $|\mathrm{K}^{(\sigma)}_{b,\beta}(x,y)| \le C\exp\left(-\frac{\beta_1}{\sigma}y+\frac{b_m}{\sigma}x\right)$, (Lemma 3.10)

   Assume here that $b_1 \le b_2 \le \dots \le b_m < \beta_1 \le \beta_2 \dots \le \beta_M$. Again, Lemma 3.10 was cited from [2, Lemma B.4], where it was stated for $\beta$ and $b$ being real numbers and not vectors. Now $b_1,\dots,b_{m-1}$ and $\beta_2,\dots,\beta_M$ are on the appropriate sides of $b_m$ and $\beta_1$ thus one can give basically the same reasoning as in [2]. The precise statement for this case can be found below in Lemma 4.1.

   (c) Hadamard bound (Lemma 3.11) and Lebesgue

   Having the same upper bound for the kernel, one can apply the Hadamard bound and Lebesgue's dominated convergence theorem, and get a summable sequence in the same way as in the proof of Proposition 3.12.

   Hence the proof of Step 2, that is the proof of the Fredholm determinants' convergence, is done.

3. $\lim_{T\to\infty}\mathbb{P}\left(\frac{\mathcal{F}^{\sigma b,\sigma\beta}(T,0)+T/24}{(T/2)^{1/3}} \le r\right) = F_{BP,b,\beta}(r)$ (Theorem 3.9)

   Proof of Step 3, i.e. of Theorem 2.9:

   With $S=e^{-r/\sigma}$, $\sigma=(2/T)^{1/3}$ and $\Theta_T(x)=\exp(e^{-r/\sigma})$, Step 1. and Step 2. imply

$$\mathbb{E}\left[\Theta_T\left(\frac{\mathcal{F}^{\sigma b,\sigma\beta}(T,0)+T/24}{\sigma^{-1}}-r\right)\right] = \mathbb{E}\left[\exp(-Se^{\mathcal{F}^{\sigma b,\sigma\beta}(T,0)+T/24})\right] =$$

$$= \det\left(\mathbb{1}-\mathrm{K}^{(\sigma)}_{\sigma b,\sigma\beta}\right)_{L^2(\mathbb{R}_+)} \to \det\left(\mathbb{1}-\mathrm{K}_{BP,b,\beta}\right)_{L^2(r,\infty)} = F_{BP,b,\beta}(r) \quad \text{as } \sigma\to 0.$$

   Then by Lemma 3.13 Step 3. immediately follows with the same reasoning as in Theorem 3.9.

□

Now let us state the lemma we referred to in Step 2.(b).

**Lemma 4.1.** *[2, Lemma B.4] Fix* $b_1 \le b_2 \le \cdots \le b_m < \beta_1 \le \beta_2 \cdots \le \beta_M$ *so that* $\beta_i - b_j < 1$ *for any* $1 \le i \le M$ *and* $1 \le j \le m$*. Then there is a finite constant* $C$ *such that for any* $x, y \in \mathbb{R}_+$

$$|\mathrm{K}^{(\sigma)}_{b,\beta}(x,y)| \le C \exp\left(-\frac{\beta_1}{\sigma} y + \frac{b_m}{\sigma} x\right), \tag{4.1.1}$$

*see* $\mathrm{K}^{(\sigma)}_{b,\beta}$ *in Definition 3.3 and* $\sigma$ *is given by* (3.2.1).

## 4.2 Scaling limit

The only statement that is still needed to make complete the proof of Theorem 2.9 is the one that determines the scaling limit of the semi-discrete partition function $\mathbf{Z}^{a,\alpha}$. First we make clear the scaling of the semi-discrete polymer's parameters similarly to the $m = M = 1$ case.

Fix $T > 0$, $X \in \mathbb{R}$ and real vectors $b = (b_1, \ldots, b_m, 0, \ldots, 0) \in \mathbb{R}^N$ and $\beta = (\beta_1, \ldots \beta_M) \in \mathbb{R}^M_+$. Recall the definition of $\theta_\kappa$ from Definition 3.5. Now let $a$ and $\alpha$ be scaled in the following way:

$$\begin{aligned} a_j = &\quad \vartheta + b_j, \quad j = 1, 2, \ldots, N \\ \alpha_i = &\quad \vartheta + \beta_i, \quad i = 1, 2, \ldots, M \end{aligned} \tag{4.2.1}$$

where $\vartheta = \theta_{\sqrt{T/N}} \simeq \sqrt{N/T} + \frac{1}{2}$. Now before defining the scaling factor and stating the theorem, we give a heuristic explanation for the order of magnitude of the scaling factor. It shows that the scaling factor indicates the relation between $\mathbf{Z}^{a,\alpha}$ and (the also semi-discrete) $\mathbf{Z}^{b,\beta}$.

Let $B_1, \ldots, B_N$ be independent Brownian motions with drifts $a_1, \ldots, a_N$ respectively and let $B^{(1)}, \ldots, B^{(N)}$ be independent standard Brownian motions. Let

$\omega_{-i,j} \sim -\ln\Gamma(\alpha_i - a_j)$, and let $\tau = \sqrt{NT} + X$. Then by (2.1.1)

$$E^{(a,\alpha)}(\phi) = \sum_{\substack{(-i,j)\in\phi^d \\ j\le m}} \omega_{-i,j} + \sum_{\substack{(-i,j)\in\phi^d \\ j>m}} \omega_{-i,j} + B_n(s_n)$$

$$+\big(B_{n+1}(s_{n+1}) - B_{n+1}(s_n)\big) + \ldots + \big(B_N(\tau) - B_N(s_{N-1})\big) =$$

$$= \sum_{\substack{(-i,j)\in\phi^d \\ j\le m}} \omega_{-i,j} + \sum_{\substack{(-i,j)\in\phi^d \\ j>m}} \omega_{-i,j} + B^{(n)}(s_n)$$

$$+\big(B^{(n+1)}(s_{n+1}) - B^{(n+1)}(s_n)\big) + \ldots + \big(B^{(N)}(\tau) - B^{N}(s_{N-1})\big)$$

$$+b_n s_n + b_{n+1}(s_{n+1} - s_n) + \ldots b_m(s_m - s_{m-1})$$

$$+\vartheta\left(s_n + s_{n+1} - s_n + \ldots + \tau - s_{N-1}\right).$$

Now for $j \le m$ it is true that $\omega_{-i,j} \sim -\ln\Gamma(\beta_i - b_j)$, because $\beta_i - b_j = \alpha_i - a_j$. We also know $\omega_{-i,j} \sim -\ln\Gamma(\alpha_i)$ for $j > m$, because $a_j = 0$ for $j > m$. But then $\omega_{-i,j} \simeq -\ln\sqrt{\frac{N}{T}}$, if $N$ is large, because of the definition of $\alpha$. Furthermore, the standard Brownian motion terms, and the terms with coefficients $b_k$ ($k = n, \ldots, m$) give together the sum of the increments of Brownian motions with drifts $b_1, \ldots b_m, 0 \ldots 0$. Finally the last term is just $\vartheta\tau = \vartheta(\sqrt{TN} + X)$ which can be factored out of the integral and sums defining the semi-discrete partition function. Let us denote informally by $\overset{d}{\simeq}$ that the order of magnitude of the LHS and of the RHS is the same in distribution. Then by the argument above, we have

$$\mathbf{Z}^{a,\alpha}\left(\sqrt{TN} + X, N\right) \overset{d}{\simeq}$$

$$\exp\left((N-m)\left(-\ln\left(\sqrt{\frac{N}{T}}\right)\right) + \vartheta\left(\sqrt{TN} + X\right)\right)\mathbf{Z}^{b,\beta}\left(\sqrt{TN} + X, N\right)$$

and thus the scaling constant should be

$$C(N,m,T,X) = \exp\left((N-m)\left(-\ln\left(\sqrt{\frac{N}{T}}\right)\right) + \vartheta\left(\sqrt{TN} + X\right)\right) =$$

$$\exp\left(\frac{1}{2}(N-m)\ln\left(\frac{T}{N}\right) + N + \frac{1}{2}\left(\sqrt{TN} + X\right) + X\sqrt{\frac{N}{T}}\right). \tag{4.2.2}$$

Now we state the theorem which was needed to prove Step 1 in the proof of Theorem 2.9. The initial condition is the one given in Definition 2.7, so here can be seen the importance of that pretty special construction.

**Theorem 4.2.** *[19] Fix $T > 0$, $X \in \mathbb{R}$ and real vectors $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $\beta = (\beta_1, \dots \beta_M) \in \mathbb{R}_+^M$ such that $b_j < \beta_i$ for all $1 \leq j \leq m$ and $1 \leq i \leq M$. Consider the semi-discrete directed random polymer with parameters $a$ and $\alpha$ defined in (4.2.1) (let $b_{m+1} = \cdots = b_N = 0$ at this point) and with partition function $\mathbf{Z}^{a,\alpha}(\tau, N)$ form Definition 2.1. The scaling factor $C(N, m, T, X)$ is given by (4.2.2). Then, as $N$ goes to infinity,*

$$\frac{\mathbf{Z}^{a,\alpha}(\sqrt{TN} + X, N)}{C(N, m, T, X)} \Rightarrow \mathcal{Z}^{b,\beta}(T, X).$$

*The convergence is in distribution and $\mathcal{Z}^{b,\beta}(T, X)$ is the solution to the SHE with initial data $Z_0^{b,\beta}(X)$, see Definition 2.7.*

## 4.3 Conclusion

Semi-discrete and continuum directed random polymer models were investigated. We summarized many important known results in the course of our discussion. A special initial condition was made clear which was needed to the scaling limit of the semi-discrete partition function. We gave a Fredholm determinant formula for the Laplace transform of the continuous partition function (which is also a Fredholm determinant formula for a double exponential expression of the KPZ solution free energy). With the help of this formula, we found the limiting distribution of the free energy fluctuations at large scale. In other words, we determined the large time limit behavior of the Hopf-Cole solution of the KPZ equation for a particular initial data. The limiting distribution was the Borodin-Péché distribution, which was given by a generalized Airy kernel and which was derived as a scaling limit of a last passage percolation model. Since the scaling factor was $T^{1/3}$ and due to the above properties of the Borodin-Péché distribution, our statement was in accordance with the KPZ universality conjecture.

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