Boundedness of the objective function

Combinatorial Optimization – Group K

Class 14

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Let A be an $m \times n$ matrix, b a column vector of dimension m and c a row vector of dimension n. Even if the solvability of $Ax \leq b$ is already known, to make the linear programming problem $\max\{cx\colon Ax \leq b\}$ sensible, we still need to ensure that the objective function cx is bounded from above on the set of solutions of $Ax \leq b$. The following theorem gives necessary and sufficient conditions on this question.

Theorem („Three-Cage Theorem"). *Assume that $Ax \leq b$ is solvable. Then the following statements are equivalent:*

(1) cx is bounded from above on $\{x\colon Ax \leq b\}$;

(2) the system $yA = c$, $y \geq 0$ is solvable;

(3) the system $Az \leq 0$, $cz > 0$ is not solvable.

Proof. We first show that *(1)* implies *(3)*. Let x_0 be a solution of $Ax \leq b$ and suppose by way of contradiction that there exists a solution z of $Az \leq 0$, $cz > 0$. Then for all $\lambda \geq 0$ the vector $x_\lambda = x_0 + \lambda z$ is a solution of $Ax \leq b$ since

$$Ax_\lambda = A(x_0 + \lambda z) = Ax_0 + \lambda(Az) \leq Ax_0 + 0 \leq b.$$

On the other hand, $cx_\lambda = c(x_0 + \lambda z) = cx_0 + \lambda(cz)$ which implies by $cz > 0$ that cx_λ can be made arbitrarily large by a suitable choice of λ. This contradicts *(1)*.

The equivalence of *(2)* and *(3)* is an easy corollary of the second form of the Farkas-lemma: apply it on A^T (the transpose of A) and in case of *(3)* take the negative of a solution.

Finally, we show that *(2)* implies *(1)*. Let y be a solution of $yA = c$, $y \geq 0$ and let x be an arbitrary solution of $Ax \leq b$. Then

$$cx = (yA)x = y(Ax) \leq yb$$

(where the last inequality follows from $Ax \leq b$ and $y \geq 0$). Hence yb is an upper bound on the value of cx on the set of solutions of $Ax \leq b$, so *(1)* holds.

We have shown the implications *(1)* $\Longrightarrow$ *(3)* $\Longrightarrow$ *(2)* $\Longrightarrow$ *(1)* which concludes the proof. □