The duality theorem

Combinatorial Optimization – Group K

Class 15

Spring 2023

1. (a) Find the dual of the following linear program.
   (b) Show that $x_1 = 3$, $x_2 = -1$, $x_3 = 0$ is an optimal solution of the primal program and $y_1 = 4$, $y_2 = 2$, $y_3 = 3$, $y_4 = 0$ is an optimal solution of the dual program.

$$\begin{aligned}
&\max\{17x_1 + 17x_2 + 17x_3\}\\
&\text{subject to}\\
&x_1 + 2x_2 + 3x_3 \le 1\\
&2x_1 + 3x_2 + x_3 \le 3\\
&3x_1 + x_2 + x_3 \le 8\\
&2x_1 + 5x_2 \le 2
\end{aligned}$$

2. (a) Find the dual of the following linear program.
   (b) Determine the minimum value of the (primal) program.

$$\begin{aligned}
&\min\{4x_1 + 10x_2 + 6x_3 + 5x_4\}\\
&\text{subject to}\\
&4x_1 - 5x_2 + 3x_3 + 5x_4 \le 5\\
&2x_1 - x_2 - x_3 + x_4 \le 1\\
&3x_1 + 2x_2 + 2x_3 + 3x_4 \ge -2
\end{aligned}$$

3. (a) Find the primal program whose dual is the following linear program.
   (b) Determine the maximum value of the primal program.

$$\begin{aligned}
&\min\{y_1 + y_2\}\\
&\text{subject to}\\
&y_1 + 3y_2 = 5\\
&2y_1 - 5y_2 = -1\\
&3y_1 - y_2 = 5\\
&y_1 \ge 0,\ \ y_2 \ge 0
\end{aligned}$$

4. Using an LP solver, we found that $x_1 = 0.3$, $x_2 = 0.1$, $x_3 = 1$ is an optimal solution of the following linear program.
   (a) Find the dual of the linear program.
   (b) Decide if the dual is solvable. If yes, then decide whether the objective function of the dual is bounded from below on the set of its solutions. If yes, then determine the minimum value of the dual program.

$$\begin{aligned}
&\max\{4x_1 + 2x_2 + 3x_3\}\\
&\text{subject to}\\
&8x_1 + 6x_2 - x_3 \le 2\\
&3x_1 + x_2 + 4x_3 \le 5\\
&5x_1 + 4x_2 + x_3 \le 3\\
&x_1 - 3x_2 + x_3 \ge 1
\end{aligned}$$

5. Assume that the system of linear inequalities $Ax \leq b$ is solvable and cx is bounded from above on its set of solutions. Assume further that the system of linear equations $Ax = b$ is also solvable and x_0 is one of its solutions. Show that x_0 is an optimal solution of the linear program $\max\{cx\colon Ax \leq b\}$.

6. Decide if the following statements are true or false.

   (a) If the primal problem is solvable, then so is the dual.

   (b) If the dual problem is solvable, then so is the primal.

   (c) At least one of the primal and the dual problems is solvable.