The duality theorem

Combinatorial Optimization – Group K

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The notion of duality in linear programming

As before, let A be an $m \times n$ matrix, b a column vector of dimension m and c a row vector of dimension n.

In condition *(3)* of the „Three-Cage Theorem", a new system of linear inequalities appears: $yA = c$, $y \geq 0$. Moreover, we also obtained the following:

if y is a solution of this system and x is a solution of the original system $Ax \leq b$, then $cx \leq yb$. $(*)$

Therefore, to obtain a best possible upper bound on the value of $\max\{cx\colon Ax \leq b\}$, it is obviously worth looking for a solution y of $yA = c$, $y \geq 0$ for which yb is as small as possible. This means that a new linear programming problem has materialized itself:

$$\min\{yb\colon yA = c, y \geq 0\}.$$

This linear program is called the *dual* of the original program $\max\{cx\colon Ax \leq b\}$, which, in return, is referred to as the *primal program* to avoid ambiguity. (It is worth observing that the dual problem contains minimization instead of maximization, the roles of b and c are swapped, A is replaced by its transpose and instead of a system of inequalities we look for a nonnegative solution of a system of linear equations.)

It is useful to have another look at the „Three-Cage Theorem" and draw its conclusions in terms of the relation between the primal and the dual programs. Assuming that the primal program is solvable and its objective function is bounded from above on its set of solutions it follows that

(1) the dual program is also solvable;

(2) the objective function of the dual program is bounded from below on its set of solutions;

(3) $\max\{cx\colon Ax \leq b\} \leq \min\{yb\colon yA = c, y \geq 0\}$.

Here *(1)* is a direct corollary of the „Three-Cage Theorem", and *(2)* and *(3)* follow from $(*)$: cx is a lower bound for the minimum of the dual for any solution x of the primal and $cx \leq yb$ for any pair of solutions implies that the same relation holds for the two optima.

Actually, much more is true.

The duality theorem

The *Duality Theorem of Linear Programming*, a theorem of utmost importance in the theory of linear programming, claims that inequality can be replaced by equation in *(3)*:

$$\max\{cx\colon Ax \leq b\} = \min\{yb\colon yA = c, y \geq 0\}.$$

Without its proof, we state the duality theorem in its full extent.

Theorem (Duality Theorem of Linear Programming)**.** *Assume that a primal program* $\max\{cx\colon Ax \leq b\}$ *is given such that* $Ax \leq b$ *is solvable and* $\{cx\colon Ax \leq b\}$ *is bounded from above. Then the following are true:*

(1) the dual system $yA = c$, $y \geq 0$ *is also solvable;*

(2) the objective function of the dual program is bounded from below on its set of solutions;

(3) the primal program attains its maximum and the dual attains its minimum;

(4) $\max\{cx\colon Ax \leq b\} = \min\{yb\colon yA = c, y \geq 0\}$.