The duality theorem – Form 2

Combinatorial Optimization – Group K

Class 16

Spring 2023

1. (a) Find the dual of the following linear program.

   (b) Determine the maximum value of the primal program. (Hint: use the chicken feed problem of Class 5, Problem 4.)

$$\begin{array}{l} \max\{20x_1 + 20x_2 + 12x_3\} \\ \text{subject to} \\ 10x_1 + 5x_2 + 2x_3 \leq 1.1 \\ 4x_1 + 5x_2 + 6x_3 \leq 1 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

2. Determine the minimum value of the following linear program (without using a computer).

$$\begin{array}{l} \min\{40x_1 + 22x_2 + 23x_3 + 50x_4\} \\ \text{subject to} \\ x_1 + x_2 + 2x_3 + 5x_4 = 3 \\ 4x_1 + 2x_2 + x_3 + x_4 = 1 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \end{array}$$

3. Find an optimal solution both for the following linear program (with n variables) and for its dual.

$$\begin{array}{lcl} \max\{nx_1 + (n-1)x_2 + \ldots + 2x_{n-1} + x_n\} \\ \text{subject to} \\ x_1 & \leq & 1 \\ x_1 + x_2 & \leq & 2 \\ x_1 + x_2 + x_3 & \leq & 3 \\ & \vdots & \\ x_1 + x_2 + x_3 + x_n & \leq & n \\ x_1, x_2, \ldots, x_n & \geq & 0 \end{array}$$

4. The following linear program got severely injured in an unfortunate accident.

$$\begin{array}{l} \max\{\square x_1 + \square x_2 + \square x_3\} \\ \text{subject to} \\ 2x_1 + 3x_2 + 4x_3 \leq 10 \\ \square x_1 + \square x_2 + \square x_3 \leq 0 \\ \square x_1 + \square x_2 + \square x_3 \leq 0 \\ \square x_1 + \square x_2 + \square x_3 \leq 0 \end{array}$$

   The $\square$'s represent all the data that was lost in the unfortunate accident. However, we know that $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ is an optimal solution of this program. Determine the maximum value of the program.

5. Find the dual of the following linear program and try to reduce them to as simple a form as possible.

$$\begin{array}{l} \min\{10x_1 + 11x_2 + 12x_3\} \\ \text{subject to} \\ x_1 + 2x_2 + 3x_3 \leq 15 \\ 4x_1 + 5x_2 + 6x_3 \geq 20 \\ 7x_1 + 8x_2 + 9x_3 = 35 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

6. Reduce the duals of the following primal problems to the simplest form.

   (a) $\min\{yb\colon yA \geq c, y \geq 0\}$
   (b) $\max\{cx\colon Ax = b, x \geq 0\}$
   (c) $\min\{yb\colon yA = c, y \geq 0\}$

7. Let A be an $m \times n$ matrix whose first column is b. Furthermore, denote by c the row vector of dimension n whose entries are all 1's. Assume that the objective function cx is bounded from above on the set of solutions of the system of linear inequalities $Ax \leq b$. Find the maximum value of $\max\{cx\colon Ax \leq b\}$.

8. Assume that the system of linear inequalities $Ax \leq b$ is solvable and cx is bounded from above on its set of solutions. Let $M = \max\{cx\colon Ax \leq b\}$. A few weeks ago (Class 9, Problem 4), we saw that $b > 0$ does not imply $M > 0$: if $c = 0$ then, no matter what b is, $M = 0$ is obvious. However, the question was left open if this was the only counterexample. So answer the question now: does $b > 0$ and $c \neq 0$ imply $M > 0$?