The duality theorem – Form 2

Combinatorial Optimization – Group K

Class 16

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The duality theorem for linear programs with nonnegative variables

As before, let A be an $m \times n$ matrix, b a column vector of dimension m and c a row vector of dimension n.

In applications of linear programming, nonnegativity of the variables is very often assumed. That is, linear programs of the form

$$\max\{cx\colon Ax \leq b, x \geq 0\} \tag{i}$$

frequently occur. Obviously, this is equivalent to $\max\{cx\colon A'x \leq b'\}$, where

$$A' = \left(\frac{A}{-I}\right), \qquad b' = \left(\frac{b}{0}\right).$$

By definition, the dual of this problem is

$$\min\{y'b'\colon y'A' = c, y' \geq 0\}. \tag{ii}$$

It is useful to break up y' into two parts: $y' = (y \mid y_1)$, where y is m-dimensional and y_1 is n-dimensional. Then (ii) can be expressed in terms of y and y_1 as follows: $\min\{yb\colon yA - y_1 = c, y \geq 0, y_1 \geq 0\}$. It is easy to see that the role of y_1 in this program is only to ensure that the system of inequalities $yA \geq c$ holds. (Indeed, c is obtained from yA by subtracting the nonnegative vector y_1.) Therefore (ii) can be rewritten in the following simplified form:

$$\min\{yb\colon yA \geq c, y \geq 0\}. \tag{iii}$$

This simplified form is equivalent to (ii) in the sense that the value of the minimum is the same, an optimal solution y' of (ii) can be transformed into an optimal solution y of (iii) by deleting its last n elements and vice versa, an optimal solution y of (iii) can be transformed into an optimal solution y' of (ii) by setting $y_1 = yA - c$ and $y' = (y \mid y_1)$. Therefore if the primal is of the form (i), then the dual is always expressed in the form (iii). Obviously, the duality theorem is also valid for the primal-dual pair (i) and (iii) as this is simply a special case of the general theorem. In more detail, we obtained the following.

Theorem (Duality Theorem of Linear Programming – Form 2)**.**

Assume that a primal program $\max\{cx\colon Ax \leq b, x \geq 0\}$ *is given such that* $Ax \leq b, x \geq 0$ *is solvable and* $\{cx\colon Ax \leq b, x \geq 0\}$ *is bounded from above. Then the following are true:*

(1) the dual system $yA \geq c$, $y \geq 0$ *is also solvable;*

(2) the objective function of the dual program is bounded from below on its set of solutions;

(3) the primal program attains its maximum and the dual attains its minimum;

(4) $\max\{cx\colon Ax \leq b, x \geq 0\} = \min\{yb\colon yA \geq c, y \geq 0\}$.