Complementary Slackness

Combinatorial Optimization – Group K

Class 17

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As before, let A be an $m \times n$ matrix, b a column vector of dimension m and c a row vector of dimension n, and assume that the primal program $\max\{cx \colon Ax \leq b\}$ is given such that $Ax \leq b$ is solvable and cx is bounded from above on its set of solutions.

The notion of the dual linear program arose from the observation that if x is a solution of the primal and y is a solution of the dual, then $cx \leq yb$. The extra information that the duality theorem added to this is that, provided that the conditions of the duality theorem hold, there is a pair of solutions x^* and y^* for which $cx^* = y^*b$ holds (and hence the maximum of the primal equals the minimum of the dual). It is worth comparing this with the calculation that yielded the inequality $cx \leq yb$:

$$cx = (yA)x = y(Ax) \leq yb.$$

How can it happen that this inequality is fulfilled with equality by x^* and y^*? Obviously, $cx^* = y^*b$ holds if and only if $y^*(Ax^*) = y^*b$; this, in return, is equivalent to saying that for every i either the i-th element of the column vector Ax^* is equal to the i-th element of b or the i-th element of y^* is zero or both. This observation, as simple as it is, turns out to be very useful in many applications, so it is worth formulating as a theorem.

Theorem (Complementary slackness)**.** *Assume that a primal program* $\max\{cx \colon Ax \leq b\}$ *is given such that* $Ax \leq b$ *is solvable and* cx *is bounded from above on its set of solutions. Assume further that* x^* *is a solution of the primal program and* y^* *is a solution of the dual program* $\min\{yb \colon yA = c, y \geq 0\}$*. Then* x^* *and* y^* *are optimal solutions of the respective programs if and only if*

$$\forall 1 \leq i \leq m \colon \quad y^*(i) = 0 \quad \textit{or} \quad a^i x^* = b(i) \quad \textit{(or both)},$$

where $y^*(i)$ *and* $b(i)$ *denote the* i*-th elements of the corresponding vectors and* a^i *denotes the* i*-th row of the matrix* A*.*