Complementary Slackness – Form 2

Combinatorial Optimization – Group K

Class 18

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As before, let A be an $m \times n$ matrix, b a column vector of dimension m and c a row vector of dimension n, and assume that the primal program $\max\{cx\colon Ax \le b, x \ge 0\}$ is given such that $Ax \le b$, $x \ge 0$ is solvable and cx is bounded from above on its set of solutions.
If x is a solution of the primal and y is a solution of the dual, then

$$cx = (yA)x = y(Ax) \le yb.$$

By the duality theorem, there is a pair of solutions x^* and y^* for which $cx^* = y^*b$ holds. Hence $cx^* = y^*b$ for a pair of solutions x^* and y^* amounts to saying that $cx^* = (y^*A)x^*$ and $y^*(Ax^*) = y^*b$. The following theorem is nothing but an expansion on this.

Theorem (Complementary slackness – Form 2)**.** *Assume that a primal program* $\max\{cx\colon Ax \le b,\ x \ge 0\}$ *is given such that* $Ax \le b$*,* $x \ge 0$ *is solvable and* cx *is bounded from above on its set of solutions. Assume further that* x^* *is a solution of the primal program and* y^* *is a solution of the dual program* $\min\{yb\colon yA \ge c, y \ge 0\}$*. Then* x^* *and* y^* *are optimal solutions of the respective programs if and only if*

$$\forall 1 \le i \le m\colon \quad y^*(i) = 0 \quad \textit{or} \quad a^i x^* = b(i) \quad \textit{(or both),}$$

and

$$\forall 1 \le j \le n\colon \quad x^*(j) = 0 \quad \textit{or} \quad y^* a_j = c(j) \quad \textit{(or both),}$$

where a^i *denotes the* i*-th row and* a_j *denotes the* j*-th column of* A*.*