The Maximum Weight Bipartite Matching Problem

Combinatorial Optimization – Group K

Class 23

Spring 2023

1. Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$ and assume that for every $1 \le i \le 4$ and $1 \le j \le 5$, the vertex a_i is adjacent to b_j in the bipartite graph $G = (A, B; E)$. Furthermore, let the weight of the edge $\{a_i, b_j\}$ be the entry in the intersection of the i-th row and the j-th column of the matrix on the left below for every i and j.

$$\begin{pmatrix} 4 & 5 & 5 & 7 & 3 \\ 3 & 5 & 3 & 6 & 3 \\ 2 & 6 & 6 & 6 & 4 \\ 3 & 5 & 5 & 7 & 3 \end{pmatrix}$$

   (a) For what values of the parameter p is it true that the mapping c shown in the table below is a labeling?

   (b) Does there exist a value of p for which the mapping c is of minimum sum among all non-negative valued labelings of G?

   (c) Does there exist a value of p for which the mapping c is of minimum sum among all arbitrary (real) valued labelings of G?

v	:	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	b_5
$c(v)$	:	4	3	4	p	0	2	2	3	0

2. Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$ and assume that for every $1 \le i, j \le 4$ the vertex a_i is adjacent to b_j in the bipartite graph $G = (A, B; E)$. Furthermore, let the weight of the edge $\{a_i, b_j\}$ be the entry in the intersection of the i-th row and the j-th column of the matrix on the right for every $1 \le i, j \le 4$.

$$\begin{pmatrix} 7 & 4 & 5 & 0 \\ 5 & 3 & 2 & 0 \\ 4 & 3 & 3 & -1 \\ 4 & 2 & 2 & -1 \end{pmatrix}$$

   (a) Is it true that the values given in the following table form a labeling?

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
2	0	0	-1	5	3	3	0

   (b) Prove that the edges corresponding to the main diagonal (that is $\{a_1, b_1\}$, $\{a_2, b_2\}$, $\{a_3, b_3\}$, $\{a_4, b_4\}$) form a maximum weight perfect matching in G.

   (c) Find a maximum weight matching in the same graph.

3. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$ and assume that for every $1 \le i, j \le 5$, the vertex a_i is adjacent to b_j in the bipartite graph $G = (A, B; E)$. Furthermore, let the weight of the edge $\{a_i, b_j\}$ be the entry in the intersection of the i-th row and the j-th column of the matrix below for every $1 \le i, j \le 5$. Is it true that the values given in the table below form a minimum sum labeling (that is, a labeling such that the sum of the labels is minimum)?

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 4 & 3 & 3 & 4 & 2 \\ 5 & 5 & 6 & 5 & 5 \\ 6 & 6 & 6 & 6 & 5 \\ 6 & 5 & 7 & 6 & 4 \end{pmatrix}$$

a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3	b_4	b_5
2	4	6	6	6	0	0	1	1	-1

4. Let $A = \{a_1, a_2, \ldots, a_k\}$, $B = \{b_1, b_2, \ldots, b_k\}$ and assume that for every $1 \le i, j \le k$ the edge $\{a_i, b_j\}$ is present in the bipartite graph $G = (A, B; E)$ with a weight of $i^2 + j^3$. Find a maximum weight perfect matching in G (and prove that it is maximum).