The Maximum Weight Bipartite Matching Problem

Combinatorial Optimization – Group K

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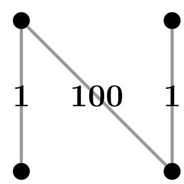
The Maximum Weight Bipartite Matching Problem

Previously we defined and gave an efficient algorithm for the Maximum Bipartite Matching Problem; now we consider a far-reaching generalization of this. The *Maximum Weight Bipartite Matching Problem* is defined as follows: given a bipartite graph $G = (A, B; E)$ and a weight function $w\colon E \to \mathbb{R}$ on its set of edges, a matching M is sought for which $\sum_{e \in E} w(e)$ is maximum possible.

The Optimum Assignment Problem

Assume that a bipartite graph $G = (A, B; E)$ is given that is known to have at least one perfect matching. Furthermore, a weight function $w\colon E \to \mathbb{R}$ is also given. Then the *Optimum Assignment Problem* is defined as follows: find a perfect matching M for which $\sum_{e \in E} w(e)$ is maximum possible.

Apparently, the only difference between the Maximum Weight Bipartite Matching Problem and the Optimum Assignment Problem is the requirement of a perfect matching. Since a perfect matching is a special matching, the maximum of the latter problem is smaller than or equal to that of the former one. However, equality is not true in general, not even if the graph has a perfect matching: the maximum weight of a matching and a perfect matching is 100 and 2, respectively, in the graph of the following figure.



A Useful Reduction

Later on, we give a combinatorial algorithm for the Optimum Assigmnent problem but first we show how the Maximum Weight Bipartite Matching Problem can be reduced to the Optimum Assignment problem; hence the above mentioned combinatorial algorithm is also usable for the Maximum Weight Bipartite Matching Problem.

So assume that an instance of the latter problem, a bipartite graph $G = (A, B; E)$ and a weight function $w\colon E \to \mathbb{R}$ are given. Perform the following modifications on the input:

1. remove all edges with negative weight from G;
2. if $|A| \neq |B|$, then add the required number of extra vertices to A or B to make them equal in size;
3. whenever a pair of vertices $a \in A$, $b \in B$ is not adjacent, connect them by an extra edge and take the weight of the extra edge to be 0.

Denote the obtained graph and the obtained weight function by G' and w', respectively and assume that M' is a maximum weight perfect matching in G'. (Clearly, G' has a perfect matching.) Then the matching M obtained from M' by deleting all its extra edges (added in Step 2 above) is of maximum weight in G. Indeed, if G had a matching M^* with a higher total weight than M, then arbitrarily pairing all vertices in G' not covered by M^* (along edges of G') would yield a perfect matching with a higher total weight than M' in G', a contradiction.

Labelings

The main tool of the above mentioned algorithm for the Optimum Assignment Problem is the notion of a labeling: the optimality of the algorithm (that is, a proof for the fact that the perfect matching the algorithm terminates with is of maximum weight) is guaranteed by a labeling.

Definition. For a bipartite graph $G = (A, B; E)$ and weight function $w\colon E \to \mathbb{R}$, the assignment $c\colon (A \cup B) \to \mathbb{R}$ is called a *labeling* if $c(a) + c(b) \geq w(e)$ holds for each edge $e = \{a, b\}$.

We mention that viewing the Optimum Assignment Problem problem as a linear programming problem also offers the possibility of proving the following theorem by applying the duality theorem on it. This is indeed doable, but we omit the details here due to lack of space and time. Instead, we give a simple combinatorial proof for it later.

Theorem (Jenő Egerváry, 1931)**.** Assume that a bipartite graph $G = (A, B; E)$ and a weight function $w\colon E \to \mathbb{R}$ are given and G is known to have a perfect matching. Then the maximum weight of a perfect matching in G is equal to the minimum value of $\sum_{v \in (A \cup B)} c(v)$, where c ranges over all real-valued labelings.

Then for the Maximum Weight Bipartite Matching Problem, we can obtain the following (but we omit the details here).

Theorem (Jenő Egerváry, 1931)**.** Assume that a bipartite graph $G = (A, B; E)$ and a weight function $w\colon E \to \mathbb{R}$ are given. Then the maximum weight of a matching in G is equal to the minimum value of $\sum_{v \in (A \cup B)} c(v)$, where c ranges over all nonnegative labelings.

Note that the above theorem differs from the previous version at two points: it involves general matchings instead of perfect ones and the corresponding labeling is restricted to be nonnegative valued.

A Useful Lemma

The statement of the following lemma could easily be obtained from an LP-based analysis of the Optimum Assignment Problem by applying complementary slackness on the LP formalization of the problem and its dual. However, we now give another (simple) proof for it to emphasize that the algorithm to be presented is independent from the theory of linear and integer programming.

Lemma. Assume that a bipartite graph $G = (A, B; E)$ and a weight function $w\colon E \to \mathbb{R}$ are given. Assume further that a perfect matching M^* and a labeling c are also given in G such that the stronger condition $c(a) + c(b) = w(e)$ holds for every edge $e = \{a, b\}$ that belongs to M^*. Then M^* is a perfect matching of maximum weight.

Proof. Let M be any perfect matching in G. Then

$$\sum_{e=\{a,b\} \in M} w(e) \leq \sum_{e=\{a,b\} \in M} \big(c(a) + c(b)\big) = \sum_{v \in A \cup B} c(v),$$

where the inequality follows from the definition of a labeling (applied on all edges of M) and the equality follows from the fact that a perfect matching covers each vertex of G exactly once.

However, since $c(a) + c(b) = w(e)$ holds for edges of M^*, the same calculation repeated on M^* gives equality:

$$\sum_{e=\{a,b\} \in M^*} w(e) = \sum_{e=\{a,b\} \in M^*} \big(c(a) + c(b)\big) = \sum_{v \in A \cup B} c(v).$$

Therefore $\sum_{v \in A \cup B} c(v)$ is an upper bound on the total weight of any perfect matching that is attained by M^*, which shows that M^* is indeed of maximum weight. □

For further reference, call an edge $e = \{a, b\}$ *binding with respect to a labeling* c if $c(a) + c(b) = w(e)$ holds for e. As mentioned above, the algorithm to be presented on the Optimum Assignment Problem relies on the above lemma: it terminates with a labeling c and a perfect matching M such that each edge of M is binding with respect to c; then the above lemma guarantees that M is of maximum weight.