ON THE RELATIVE LENGTHS OF SIDES OF CONVEX POLYGONS

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Abstract. Let C be a convex body. By the relative distance of points p and q we mean the ratio of the Euclidean distance of p and q to the half of the Euclidean length of a longest chord of C parallel to pq. The aim of the paper is to find upper bounds for the minimum of the relative lengths of the sides of convex hexagons and heptagons.

1. Introduction

Let pq be the closed segment with endpoints p and q in the Euclidean plane E^2. Denote by $|pq|$ the Euclidean length of pq. Take a convex body $C \subset E^2$. Consider a chord $p'q'$ of C such that there is no longer chord in C parallel to pq. The ratio of $|pq|$ to $\frac{1}{2}|p'q'|$ is called the *C-distance of p and q* and it is denoted by $d_C(p,q)$. By the *C-length* of a closed segment we mean the C-distance of its endpoints. If it is obvious which convex body we refer to, we may use the names *relative distance* or *relative length*. We call a side of a convex n-gon *relatively short* (respectively, *relatively long*) if its relative length is not greater (not smaller) than the relative length of a side of the regular n-gon.

Doliwka and Lassak [2] proved that *every convex pentagon has a relatively short and a relatively long side*. As the relative length of the sides of a regular hexagon is 1, the analogous qustion about hexagons is whether every convex hexagon has a side of relative length at least 1, and a side of relative length at most 1. Doliwka and Lassak [2] presented the following examples that the answer for the above question is negative.

Consider the hexagon H_0 which is the convex hull of a regular triangle and its homothetic copy with the homothety centre in the centre of gravity of the triangle with the homothety ratio $1-\sqrt{3}$. The relative length of the sides of this hexagon is $8-4\sqrt{3} \approx 1.071 > 1$. The mentioned authors conjecture that every convex hexagon has a relatively long side and a side of relative length at most $8-4\sqrt{3}$. The first part of this conjecture follows from the paper [1] of Chakerian and Talley. The aim of the present paper is to prove the second part of this conjecture. We also show that every convex heptagon has a side of unit relative length.

2. Hexagons

Theorem 1. *Every convex hexagon has a side of relative length at most* $8-4\sqrt{3}$.

Our proof of this theorem is based on two lemmas.

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Lemma 1. *Let G be a convex n-gon, where $n \geq 6$. Assume that a triangle of the largest possible area inscribed in G has a side which coincides with a side of G. Then G has a side of G-length at most* 1.

Proof. Let $T = abc$ be a triangle mentioned in the formulation of our lemma. Observe that we can assume that ab is a side of G, and c is a vertex of G. At least one of the two pieces of the boundary of G between a and c contains at least two additional vertices e and f of G. For instance, let e be between c and f on this piece.

Since the ratio of the areas of two figures does not change under affine transformations, we may assume in our proof that abc is an isosceles triangle with right angle at b. Take the point d such that $S = abcd$ is a square. Since abc is a triangle of maximal area, we conclude that e and f belong to S.

Consider the convex pentagon $P = abcef$. First, we intend to show that at least one of the relative distances $d_P(c,e)$, $d_P(e,f)$, $d_P(f,a)$ is at most 1. We dissect S into four equal squares S_a, S_b, S_c, S_d containing a, b, c, d, respectively. Since G is convex, e and f are not in the interior of T. If $d_P(c,e) > 1$ and $d_P(f,a) > 1$, then $e \notin S_c$, and $f \notin S_a$ and thus $e \in S_d$ and $f \in S_d$. Hence $d_P(e,f) \leq 1$. We see that at least one of the numbers $d_P(c,e)$, $d_P(e,f)$, $d_P(f,a)$ is at most 1.

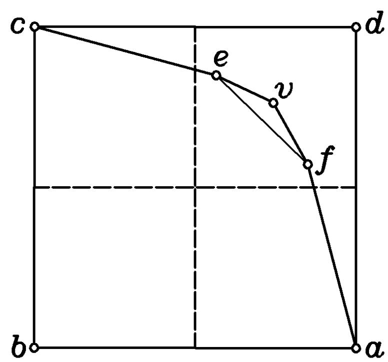

Figure 1

Finally, we intend to show that if one of the mentioned P-distances is at most 1, then G has a side of G-length at most 1. We assume that $d_P(e,f) \leq 1$ (analogical consideration can be applied for the remaining two cases). Examine the case when e and f are consecutive vertices of G. Since P is a subset of G, we have $d_G(p,q) \leq d_P(p,q)$ for arbitrary points p, q. Thus, in this case the thesis of our lemma holds true. Take into account the opposite case, when e and f are not consecutive vertices, and take a vertex v of G between them. Let V be a side of G with endpoint v. Consider the chords C_a and C_c of G with endpoints a and c, respectively, which are parallel to V. Observe that C_a or C_c is at least twice as long as V. Hence, the G-length of V is at most 1.

$\square$

Lemma 2. *Consider a convex hexagon $H = abcdef$ such that the triangle ace is regular. Let us take the lines through a, c, e parallel to the segments ce, ea, ac, respectively. The intersections of these lines are denoted by a_0, c_0, e_0 (they are opposite to a, c, e, respectively). Assume that b, d, f are in the triangle $a_0c_0e_0$ and that the angles $cab\angle, acb\angle, aef\angle, eaf\angle$ are equal α. Denote the angle $ecd\angle$ by β, and denote the angle $ced\angle$ by γ. If $0 < \alpha < \frac{\pi}{6}$, $0 < \min(\beta,\gamma) < \frac{\pi}{6}$, $d_H(c,d) \geq 8 - 4\sqrt{3}$, and $d_H(d,e) \geq 8 - 4\sqrt{3}$, then $\min(\beta,\gamma) \geq \alpha$ with equality if and only if $\alpha = \beta = \gamma = \pi$* [illegible]

Proof. We choose a Cartesian coordinate system such that a, c, e are $(0,0)$, $(1,\sqrt{3})$ and $(-1,\sqrt{3})$, respectively. Since $d_H(d,e) \geq 8-4\sqrt{3}$, then d is not in the interior of the homothetic copy C_1 of the quadrangle $cefa$ with the homothety ratio $-(4-2\sqrt{3})$ such that the image of c is e. Moreover, also d is not in the interior of the homothetic copy C_2 of the quadrangle $eabc$ with the homothety ratio $-(4-2\sqrt{3})$ such that the image of e is c. The boundaries of C_1 and C_2 inside of the triangle ca_0e intersect each other at one point. Denote it by d_0.

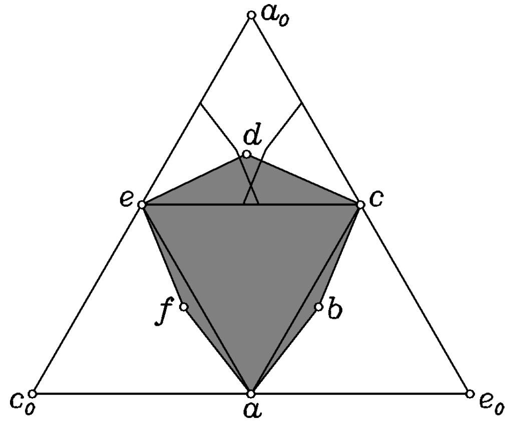

Figure 2

Case 1, when d_0 is on the images of the sides ef and bc.

The minimum of β and γ is attained for $d = d_0$. The ordinate of d_0 is

$$\tan(\alpha + \frac{\pi}{3})(7 - 4\sqrt{3}) + \sqrt{3}.$$

This implies the inequality

$$\tan(\min(\beta,\gamma)) \geq \tan(\alpha + \frac{\pi}{3})(7 - 4\sqrt{3}).$$

But it can be easily verified that if $0 < \alpha < \frac{\pi}{6}$, then $\tan(\alpha) \leq \tan(\alpha + \frac{\pi}{3})(7-4\sqrt{3})$ with equality if and only if $\alpha = \frac{\pi}{12}$. Hence, the equality can hold if and only if $\alpha = \min(\beta,\gamma) = \frac{\pi}{12}$. But when β or γ is equal to $\frac{\pi}{12}$, d_0 is the only point on the segment in the triangle eca_0 determined by the angle $\frac{\pi}{12}$ which is not in the interiors of C_1 and C_2. That is, we have $\beta = \gamma = \frac{\pi}{12}$.

Case 2: when d_0 is on the images of the sides fa and ab.

We get the minimum of β when d is d_0 or when d is the homothetic image of a in the homothetic copy of the quadrangle $cefa$. Hence $\beta \geq \frac{\pi}{6}$. A similar inequality holds for γ. Therefore $\min(\beta,\gamma) \geq \frac{\pi}{6}$, contrary to the hypothesis.

□

Proof of Theorem. Consider a convex hexagon $H = abcdef$. If a triangle of the largest possible area inscribed in H has a side which coincides with a side of H, then we apply Lemma 1.

Let us look to the opposite case when every triangle of the maximum area inscribed in H does not contain a side of H. Observe that then ace or bdf is a triangle of maximal area. Consider the first possibility (in the other one, further consideration is analogical). Since the relative distance is affine invariant, we can assume that ace is a regular triangle with vertices $a(0,0)$, $c(1,\sqrt{3})$, $e(-1,\sqrt{3})$ in a rectangular coordinate system. We provide straight lines

e parallel to the segments ce, ea, ac, respectively. Denote the point of intersection of L_c and L_e by a_0. Similarly, let c_0 be the intersection of L_a and L_e. Moreover, let e_0 be the intersection of L_a and L_c. Since ace is a triangle of maximum area inscribed in H, the points b, d and f belong to the triangle $a_0c_0e_0$. Denote the angles $cab\angle, acb\angle, ecd\angle, \ldots, eaf\angle$ by $\alpha_1, \alpha_2, \ldots \alpha_6$, respectively.

In order to prove our theorem we intend to show that if the relative lengths of the sides of H are at least $8 - 4\sqrt{3}$, then $\alpha_i = \frac{\pi}{12}$ for $i = 1, \ldots, 6$.

In further consideration we exclude the case when $\alpha_i = 0$ for a certain i because in this special situation the hexagon contains a closed segment containing three consecutive vertices which means that it has a side of relative length at most 1.

Assume that $\alpha_4 = \min\{\alpha_1, \ldots, \alpha_6\}$.

Case 1, when $\alpha_4 < \frac{\pi}{6}$.

Consider first an auxiliary hexagon H' in which we have α_4 in the place of $\alpha_1, \alpha_2, \alpha_5, \alpha_6$. Then, from $H' \subset H$ we get that $d_H(c,d) \leq d_{H'}(c,d)$ and $d_H(d,e) \leq d_{H'}(d,e)$.

Now we apply Lemma 2 for H' putting α_3 in the part of β, and α_4 in the part of α and γ. We get that $\alpha_4 \leq \alpha_4$ with equality if and only if $\alpha_3 = \alpha_4 = \frac{\pi}{12}$. Since α_4 is the minimal angle from among $\alpha_1, \ldots, \alpha_6$, all those angles are at least $\frac{\pi}{12}$. Let us take the homothetic copies of the quadrangles $cefa$ and $eabc$ with homothetic ratio $-(4 - 2\sqrt{3})$ such that the images of c and e are e and c, respectively. Since d is in the interior of neither of the two copies, we get that $\min(\alpha_1, \alpha_2)$ and $\min(\alpha_5, \alpha_6)$ are at most $\frac{\pi}{12}$. Consequently they are equal to $\frac{\pi}{12}$. Now we take an auxiliary hexagon H'' in which α_5 and α_6 are replaced by $\alpha_4 = \frac{\pi}{12}$. We apply Lemma 2 for H'' and we get that $\frac{\pi}{12} \leq \frac{\pi}{12}$ with equality if and only if $\alpha_1 = \alpha_2 = \frac{\pi}{12}$. Thus, we can apply Lemma 2 for H, and as a result we get that $\alpha_i = \frac{\pi}{12}$ for every $i \in \{i = 1, 2, \ldots, 6\}$.

It can be easily verified that this hexagon is nothing else but the hexagon H_0 mentioned in Introduction.

Case 2, when $\alpha_4 \geq \frac{\pi}{6}$.

According to our previous assumption about α_4, all the angles are at least $\frac{\pi}{6}$. Notice that in this case the area of the triangle bdf is not less than the area of the triangle ace, with equality if and only if all the six angles are $\frac{\pi}{6}$. Hence this case concerns only the regular hexagon, the relative length of the sides of which is equal to 1.

□

The proof of Theorem also shows that the only hexagons such that the relative lengths of its sides are at least $8 - 4\sqrt{3}$ are the affine images of the hexagon H_0 constructed by Doliwka and Lassak.

3. Heptagons

Corollary. *Every convex heptagon has a side of relative length at most* 1.

Proof. Let $H = abcdefg$ be a convex heptagon, such that all the relative lengths of its sides are greater than 1. According to Lemma 1 we can assume that acf is a triangle of maximal area inscribed in H. As relative distance is affine invariant, we can assume that the triangle acf is regular. Let us take a Cartesian coordinate system such that a, c, and f are $(0,0)$, $(1,\sqrt{3})$, $(-1,\sqrt{3})$, respectively. We define the points a_0, c_0, f_0 similarly like in the proof of Theorem. Since acf is a triangle of maximal area, b, d, e, g are in the triangle $a_0c_0f_0$. Let a', d', f' be the midpoints of the segments cf, a_0f and a_0c, respectively. As $d_H(c,d)$ and $d_H(e,f)$ are greater than 1, the points d and e belong to the rhombus $a'f'a_0c'$. The convexity of H implies that the slope of the segment de is between $-\sqrt{3}$ and $\sqrt{3}$. Hence $d_H(d,e) \leq 1$. But this contradicts the assumption that the relative lengths of all the sides of H are greater than 1.

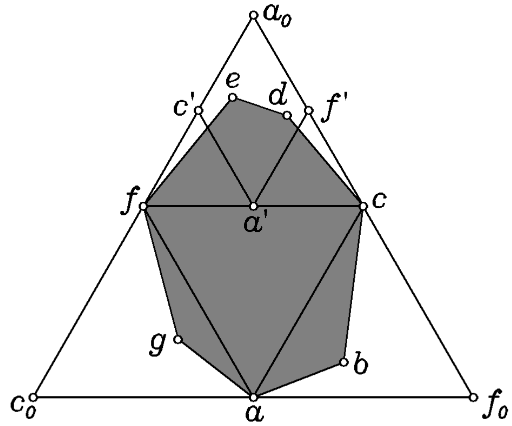

Figure 3

□

The example of the degenerated heptagon with four vertices in the vertices of a square and with three remaining vertices in the midpoints of the sides of the square shows that this result is the best possible.

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