Convex and Discrete Geometry

Relative Distance of Boundary Points of a Convex Body and Touching by Homothetical Copies

by

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Summary. The relative distance of points p and q of a convex body C is the ratio of the length of the segment pq to the half of the length of a longest chord of C parallel to pq. In this paper we find a connection between pairwise relative distances of k points in the boundary of a convex body and the ratio of k homothetical copies of the body touching it.

Let A and B be convex bodies in the Euclidean n-space E^n. If A is a subset of B, and if A contains a boundary point of B, we say that *A touches the boundary of B from inside*. If the intersection of A and B is not empty but their interiors are disjoint, we say that *A and B touch each other*. If the interiors of A and B have a common point, we call them *overlapping*.

By the *translative kissing number* $H(C)$ of a convex body $C \subset E^n$ we mean the maximal number of its mutually nonoverlapping translates touching C. Hadwiger [7] showed that for every plane convex body its translative kissing number is always at least 6 and at most 8. Grünbaum [6] proved that if C is a parallelogram, then $H(C) = 8$, and the translative kissing number of every other plane convex body is 6.

In this paper we examine the following question. Let $C \subset E^n$ be a convex body, and let t be a positive number. What is the maximal integer k such that there exist k mutually nonoverlapping homothetical copies of C with homothety ratio t touching C? From another point of view, for a certain positive integer k what is the maximal number t such that there exist k mutually nonoverlapping homothetical copies of C with homothety ratio t touching C?

To investigate this problem we establish a connection between the above homothety ratios and the C-distances of points in the boundary of a convex body. The notion of C-distance of two points is defined as below. For arbitrary points p and q of E^n and for an arbitrary convex body $C \subset E^n$ consider a chord $p'q'$ of C parallel to pq such that there is no longer chord of C parallel to pq. By the *C-distance $d_C(p,q)$ of p and q* we mean the ratio of the length of the segment pq to the half of the length of the segment $p'q'$ (see [10]). If there is no doubt about C, we also use the name *relative distance*. It is a well-known fact that the unit ball of the normed space with the norm $||x|| = d_C(x,0)$ is $\frac{1}{2}(C - C)$. Observe that for arbitrary points p and q and for every $r \in [-1,1]$ the C-distance of p and q is equal to their $[rC + (1-r)(-C)]$-distance.

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The paper [9] establishes a connection between the relative distances of k points in an arbitrary convex body C and the ratio of k positive homothetical copies of C packed into C. In [11] we see some results about positive homothetical copies touching the boundary of C from inside, and also the case when negative homothetical copies of C touch C is considered.

Theorem. *For every convex body $C \subset E^n$ and for every $t \in (0, \infty)$ the following two conditions are equivalent:*

(i) there exist k mutually nonoverlapping homothetical copies of C with homothety ratio t touching C,

(ii) there exist k points in the boundary of $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$ in pairwise C-distances at least $\frac{2t}{1+t}$.

P r o o f. First we show that (i) implies (ii).

Case 1, when $t \in (0,1)$. Let us assume that $C_1, \ldots, C_k$ are mutually nonoverlapping homothetical copies of C with homothety ratio t touching C. Denote by c_i the center of the homothety h_i which maps C into C_i, for $i = 1, \ldots, k$. Let q_i be a common point of C and C_i. As C and C_i are not overlapping, they have a common supporting hyperplane H_i containing q_i. Take the point p_i of C for which $h_i(p_i) = q_i$. Obviously, $d_C(c_i, q_i) = t d_C(c_i, p_i)$. Since there exist parallel supporting hyperplanes of C containing p_i and q_i (for instance, $h_i^{-1}(H_i)$ and H_i), we get $d_C(p_i, q_i) = 2$. That is, $t(d_C(c_i, q_i) + 2) = d_C(c_i, q_i)$. Thus, $d_C(c_i, q_i) = \frac{2t}{1-t}$. It is easy to see that for every point $s \in C$ we have $d_C(c_i, s) \geq \frac{2t}{1-t}$. Observe that the set of points whose C-distance from a point $w \in E^n$ equals to d is the boundary of the convex body $w + \frac{d}{2}(C - C)$. Hence c_i is in the boundary of $C + \frac{t}{1-t}(C - C) = \frac{1}{1-t}C + \frac{t}{1-t}(-C)$.

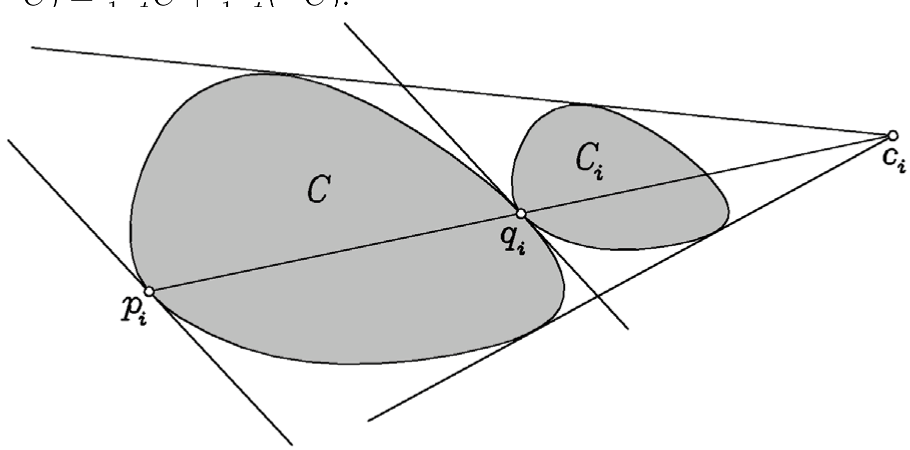

Figure 1

Now we intend to show that $d_C(c_i, c_j) \geq \frac{2t}{1-t}$ for $i, j \in \{1, \ldots, k\}$, where $i \neq j$. Let us take a point r of C. Denote $h_i(r)$ by r_i, for $i = 1, \ldots, k$. Apparently, $|rr_i| = (1-t)|rc_i|$. As the triangles rr_ir_j and rc_ic_j are similar, we conclude that $|r_ir_j| = (1-t)|c_ic_j|$. It was noted by Minkowski in [12] that for an arbitrary convex body C, if $x + C$ and $y + C$ are overlapping, touching or disjoint, then $x + \frac{1}{2}(C - C)$ and $y + \frac{1}{2}(C - C)$ are overlapping, touching or disjoint, respectively (we will apply this property a few times). Thus, as C_i and C_j are not overlapping, we obtain that $d_C(r_i, r_j) > 2t$. Hence $d_C(c_i, c_j) \geq \frac{2t}{1-t}$.

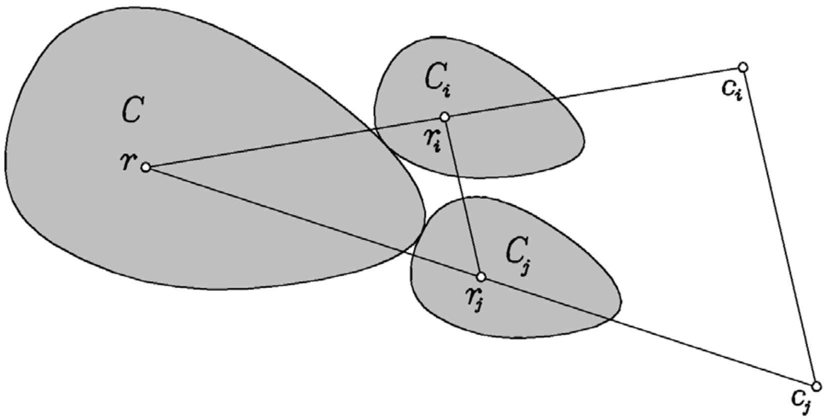

Figure 2

Finally, let us take the homothety h with the homothety ratio $\frac{1-t}{1+t}$ and with the center at the origin. Then $h(c_1), \ldots, h(c_k)$ are k points in pairwise C-distances at least $\frac{2t}{1+t}$ in the boundary of $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$.

Case 2, when $t = 1$. Let $p_1 + C, \ldots, p_k + C$ be mutually nonoverlapping translates of C touching C. Thanks to the mentioned result of [12], we see that $p_1, \ldots, p_k$ are points in C-distance 2 from the origin. Hence they are in the boundary of $C - C$. This result in [12] also implies that the pairwise C-distances of $p_1, \ldots, p_k$ are at least 2. Let h denote the homothety with the center at the origin and with the homothety ratio $\frac{1}{2}$. Then $h(p_1), \ldots, h(p_k)$ are k points in the boundary of $\frac{1}{2}C + \frac{1}{2}(-C)$ in pairwise C-distances at least 1.

Case 3, when $t \in (1, \infty)$. Let $C_1, \ldots, C_k$ be mutually nonoverlapping homothetical copies of C with homothety ratio t touching C. Denote by c_i the center of the homothety h_i that maps C into C_i, for $i = 1, \ldots, k$. We omit a consideration analogous to that in Case 1 which shows that c_i is in the boundary of $C + \frac{1}{t-1}(C - C) = \frac{t}{t-1}C + \frac{1}{t-1}(-C)$. We also omit a consideration that $d_C(c_i, c_j) \geq \frac{2t}{t-1}$ for every i, j ($1 \leq i, j \leq k$, and $i \neq j$). Let h denote the homothety with the center at the origin and with the negative homothety ratio $\frac{1-t}{1+t}$. Then $h(c_1), \ldots, h(c_k)$ are k points in the boundary of $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$ in pairwise C-distances at least $\frac{2t}{1+t}$.

Finally, observe that the considerations in all the three cases are revertible. Thus (ii) implies (i).

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Notice that our theorem remains true if we consider disjoint homothetical copies and C-distances greater than $\frac{2t}{1+t}$.

In the following part of the paper we deal only with the planar case. By $\mathcal{C}$ we mean the family of convex bodies of E^2, and by $\mathcal{M}$ we denote the family of centrally symmetric convex bodies of E^2. For every $t \in (0, \infty)$, the family of plane convex bodies which can be presented in the form $\frac{1}{1+t}C + \frac{t}{1+t}(-C)$, where all $C \in \mathcal{C}$ are taken, is denoted by $\mathcal{C}_t$.

Let t_k, where $k \geq 3$, denote the maximal number such that for every plane convex body C there exist its k mutually nonoverlapping homothetical copies with ratio t_k touching C. Analogously, let u_k, where $k \geq 5$, denote the maximal number such that there exists a plane convex body C for which there are k mutually nonoverlapping homothetical copies of C with ratio u_k touching C. Here, compactness arguments show that the above maxima exist. Obviously, both $\{t_k\}$ and $\{u_k\}$ are nonincreasing sequences. Using Theorem, we get a number of estimates for some values of t_k and u_k. These estimates are collected in the following Corollary.

Corollary. *We have* $t_5 = t_6 = 1$ *and* $\frac{1}{2} \leq t_7 \leq \frac{3}{4}$. *Moreover,* $u_5 = \frac{1}{2}(\sqrt{5}+1) \approx 1.618$, $u_6 = u_7 = u_8 = 1$, *and for every integer* $s \geq 2$ *we have* $u_{4s} = \frac{1}{s-1}$.

P r o o f. Notice that for every $t \in (0, \infty)$, the C-distance of arbitrary two points is equal to their $[\frac{t}{1+t}C + \frac{1}{1+t}(-C)]$-distance. Thus, according to Theorem, t_k is the maximal number such that the boundary of every $C \in \mathcal{C}_{t_k}$ contains k points in pairwise C-distances at least $\frac{2t_k}{1+t_k}$. Similarly, u_k is the maximal number such that there exists $C \in \mathcal{C}_{u_k}$ whose boundary contains k points in pairwise C-distances at least $\frac{2u_k}{1+u_k}$. Put $d = \frac{2t}{1+t}$. Thus $t = \frac{d}{2-d}$. Observe that $\mathcal{C}_1 = \mathcal{M}$. Furthermore, for every $t \in (0, \infty)$ we have $\mathcal{M} \subset \mathcal{C}_t \subset \mathcal{C}$. Hence, if the boundary of every plane convex body contains k points in pairwise relative distances at least d, and if there exists a centrally symmetric plane convex body whose boundary does not contain k points in pairwise relative distances greater than d, then $t_k = \frac{d}{2-d}$. Analogously, if there exists a centrally symmetric plane convex body whose boundary contains k points in pairwise relative distances at least d, and if there is no plane convex body whose boundary contains k points in pairwise relative distances greater than d, then $u_k = \frac{d}{2-d}$. We apply these two statements a few times in the remaining part of the proof.

In [1] it is proved that the boundary of every plane convex body contains five points in pairwise relative distances at least 1. It is easy to check that the boundary of the parallelogram does not contain five points in pairwise relative distances greater than 1. Therefore $t_5 = 1$.

In [3] and in [10] it is observed that the boundary of every centrally symmetric plane convex body contains six points in pairwise relative distances at least 1. As $\mathcal{C}_1 = \mathcal{M}$, we get $t_6 \geq 1$. It is also observed in [3] that there is no centrally symmetric plane convex body whose boundary contains six points in pairwise relative distances greater than 1. Consequently, our Theorem implies that there is no plane convex body which can be touched by its six mutually disjoint translates. This means that there is no convex body that can be touched by its six mutually nonoverlapping homothetical copies with homothety ratio greater than 1. Hence $u_6 \leq 1$. Obviously $t_6 \leq u_6$. Thus $t_6 = u_6 = 1$.

With respect to [8], the boundary of every plane convex body contains seven points in pairwise relative distances at least $\frac{2}{3}$. Hence $t_7 \geq \frac{1}{2}$. We omit an elementary consideration which shows that the boundary of the regular hexagon does not contain seven points in pairwise relative distances greater than $\frac{6}{7}$. This gives the estimate $t_7 \le \frac{3}{4}$.

In [2] it is proved that there exists no convex body whose boundary contains five points in pairwise relative distances greater than $\sqrt{5}-1$. The value $\sqrt{5}-1$ is attained for the regular pentagon and decagon. Therefore $u_5 = \frac{1}{2}(\sqrt{5}+1)$.

It follows from (266) on page 71 in [5] and from Theorem 2 in [4] that the circumference of every plane convex body measured in the metric $d_C(x,y)$ is at most 8. The example of the parallelogram shows that for every integer $s \ge 2$, we have $u_{4s} = \frac{1}{s-1}$. Hence $u_8 = 1$.

We see that $u_6 = u_8 = 1$. As the sequence $\{u_k\}$ is nonincreasing, we get $u_7 = 1$.

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Other values of t_k and u_k are not determined. We conjecture that there exists no plane convex body whose boundary contains 9 points in pairwise relative distances greater than $4\sin(10^\circ) \approx 0.6946$. Since this value is attained for the regular 9-gon and 18-gon, we also conjecture that $u_9 = \frac{2\sin(10^\circ)}{1-2\sin(10^\circ)}$. Moreover, we conjecture that there exists no plane convex body whose boundary contains 10 points in pairwise relative distances greater than $\frac{2}{3}$, and therefore $u_{10} = u_{11} = \frac{1}{2}$.

In the remaining part of the paper we deal with convex bodies containing their mutually nonoverlapping negative homothetical copies. Like in Theorem, we prove a connection between the ratio of the above homothetical copies and the relative distances of points in a convex body. Since the proof is analogous to the proof of Theorem, we only sketch it.

Proposition. *Let C be an arbitary convex body in E^n, and let $t \in (0,1]$. Denote by C_t the set of points of C whose C-distance from every boundary point of C is at least $\frac{2t}{1+t}$. Then the following two conditions are equivalent:*

(i) there exist k mutually nonoverlapping homothetical copies of C with homothety ratio $-t$ packed in C,

(ii) there exist k points in C_t in pairwise C-distances at least $\frac{2t}{1+t}$.

P r o o f. Consider a homothetical copy K of C with homothety ratio $-t$ packed in C. Denote by h the homothety which maps C into K, and let c be the center of homothety. For the sake of simplicity let us assume that c is the origin. Then $K = -tC$. Observe that for arbitrary sets A and B, and for arbitary $r \in [0,1]$, the set $rA + (1-r)B$ is contained in the convex hull of $A \cup B$. Therefore C contains $\frac{t}{1+t}C + \frac{1}{1+t}(-tC) = \frac{t}{1+t}(C-C)$. That is, the C-distance of c and of every boundary point of C is at least $\frac{2t}{1+t}$. So c is in C_t.

We omit a consideration analogous to that in Theorem that if $-tC$ is not contained in C, then $c \notin C_t$.

Finally, take two arbitrary homothetical copies K_1 and K_2 of C with homothety ratio $-t$. Let c_1 and c_2 be the centers of the homotheties which map C into K_1 and K_2, respectively. Similarly like in Theorem, we observe that K_1 and K_2 do not overlap if and only if $d_C(c_1, c_2) \geq \frac{2t}{1+t}$.

■

Analogously to the proof of Proposition, we can show the following. If we have k negative homothetical copies touching the boundary of C from inside, then there exist k points in the boundary of C_t in pairwise C-distances at least $\frac{2t}{1+t}$, and vica versa, if we have k points in the boundary of C_t, then there exist k negative homothetical copies touching the boundary of C from inside.

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