On the Relative Distances of Six Points in a Plane Convex Body

K. Böröczky and Z. Lángi

Abstract. Let C be a convex body in the Euclidean plane. By the relative distance of points p and q we mean the ratio of the Euclidean distance of p and q to the half of the Euclidean length of a longest chord of C parallel to pq. In this note we find the least upper bound of the minimum pairwise relative distance of six points in a plane convex body.

Denote the closed segment with endpoints p and q in the Euclidean plane E^2 by pq, and denote the Euclidean length of pq by $|pq|$. Let $C \subset E^2$ be a convex body. Take a chord $p'q'$ of C parallel to pq such that there is no longer chord of C parallel to pq. The ratio $d_C(p,q)$ of $|pq|$ to $\frac{1}{2}|p'q'|$ is called the *C-distance* of points p and q (see [5]). We also use the term *C-length* of the segment pq. If there is no doubt about C, we talk about the *relative distance* of points p and q, or the *relative length* of the segment pq. If C is centrally symmetric, the relative distance is the Minkowski distance in the Minkowski space whose unit ball is C.

It is shown by Doyle, Lagarias and Randall [2] that among six arbitrary points of an arbitrary centrally symmetric plane convex body there is a pair in relative distance at most 1. They also showed that this value is attained for every centrally symmetric convex body. For some smaller values of k, they found a connection between the minimal pairwise relative distance of k points in a centrally symmetric plane convex body on one hand, and the homothety ratio of k congruent homothetical copies of the body packed into the body on the other hand.

The paper [1] of Doliwka and Lassak implies that among five arbitrary boundary points of an arbitrary plane convex body there exists a pair in relative distance at most $\sqrt{5}-1 \approx 1.236$. In the Appendix of our paper we show that the statement remains true if we consider arbitrary points (not obligatorily boundary points) of a convex body. The examples of the regular pentagon and the regular decagon show that the value $\sqrt{5}-1$ cannot be lessened.

Lángi [3] proved that among six arbitrary boundary points of an arbitrary plane convex body there exists a pair in relative distance at most $8-4\sqrt{3} \approx 1.072$. This value is attained for the hexagon that is the convex hull of the regular triangle and its homothetical copy with center at the center of gravity of the triangle and with homothety ratio $1-\sqrt{3}$ (see [1] or [3]). This hexagon is nothing else but the convex hull of the vertices and of the midpoints of the arcs of the Reuleaux triangle. In this paper we prove the following theorem.

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Theorem. *Among six arbitrary points of an arbitrary plane convex body there exists a pair in relative distance at most* $2-\frac{2\sqrt{5}}{5}\approx 1.106$*. Furthermore, if* $p_1,\dots p_6$ *are points in a plane convex body* C *such that all their pairwise relative distances are at least* $2-\frac{2\sqrt{5}}{5}$*, then* C *is an affine regular pentagon and the points are its vertices and its center.*

Using the idea of [1] and [5] we can immediately reformulate our Theorem in the following form (see also Theorem of [4]).

No plane convex body can be packed by its six homothetical copies of ratio greater than $\frac{9}{19}-\frac{1}{19}\sqrt{5}$*. Moreover, if a plane convex body* C *can be packed by its six homothetical copies of ratio* $\frac{9}{19}-\frac{1}{19}\sqrt{5}$*, then* C *is an affine regular pentagon and the homothety centers are its vertices and its center.*

During the proof of our theorem, we denote points by small Latin letters. In a Cartesian coordinate system, the x-coordinate and the y-coordinate of a point $p\in E^2$ are denoted by p^x and p^y, respectively. We denote the straight line through the points $p,q\in E^2$ by $L(p,q)$. The value $2-\frac{2\sqrt{5}}{5}$ is denoted by λ, the value $\frac{\lambda}{2}=1-\frac{\sqrt{5}}{5}\approx 0.553$ by τ and the value $\frac{\lambda}{2-\lambda}=\sqrt{5}-1$ by ν. By the kernel of a convex pentagon P we mean the convex pentagon which is bounded by the diagonals of P.

The proof of our theorem is based on three lemmas.

Lemma 1. *Take a convex pentagon* $P=a_1a_2a_3a_4a_5$ *and take a point* p *in the kernel of* P*. Denote* $\min\{d_P(p,a_i)|i=1,\dots,5\}$ *by* $\lambda(P,p)$*. Then* $\lambda(P,p)\le\lambda$ *and equality holds if and only if* P *is an affine regular pentagon and* p *is its center.*

Proof. Compactness arguments show that the maximal value of $\lambda(P,p)$ is attained on the family of convex pentagons P and points p of the kernel of P. Moreover, if P is an affine regular pentagon and if p is its center, then $\lambda(P,p)$ is equal to λ. Hence it is enough to show that if P is not affine regular or if p is not its center, then $\lambda(P,p)$ cannot be maximal. During the proof we denote the intersection point of the line $L(p,a_i)$ and of the opposite side of P by b_i for $i=1,\dots,5$. Moreover, we denote the kernel of P by Q.

Observe that if p is a point of the boundary of Q, then $\lambda(P,p)\le 1$, which is less than λ. Thus in this case $\lambda(P,p)$ cannot be maximal. Therefore in the sequel we assume that p is in the interior of Q.

Case 1, when P has a side of P-length 2. For instance, let a_1a_2 be such a side. Instead of the condition that p is in the kernel of P, during the proof in this case we use only the terms that $p\in a_1a_3a_5$ and $p\in a_2a_3a_5$. For $i=1,\dots,5$ let us denote a maximal chord of P parallel to a_ip by u_iv_i. As $|u_iv_i|\ge|a_ib_i|$, we have $d_P(a_i,p)=\frac{|a_ip|}{\frac{1}{2}|u_iv_i|}\le\frac{2|a_ip|}{|a_ib_i|}$.

If $|a_5p|>|pb_5|$ and if $|a_3p|>|pb_3|$, then $L(a_1,a_2)$ separates p and the intersection point of $L(a_1,a_5)$ and $L(a_2,a_3)$. Thus $d_P(a_1,a_2)<2$, which is a contradiction. If $|a_3p|\le|pb_3|$, then $d_P(a_3,p)\le 1$, hence $\lambda(P,p)\le 1$. Similarly, if $|a_5p|\le|pb_5|$, then $\lambda(P,p)\le d_P(a_5,p)\le 1$.

Case 2, when P has no side of P-length 2. In this case $d_P(p, a_i) = \frac{2|a_i p|}{|a_i b_i|}$ for $i = 1, \dots, 5$.

Subcase 2.1, when P has two consecutive vertices in P-distance from p greater than $\lambda(P, p)$. Assume, for example, that $d_P(a_4, p) > \lambda(P, p)$ and that $d_P(a_5, p) > \lambda(P, p)$ (see Figure 1). For $i = 1, \dots, 5$ let us denote by H_i the open halfplane bounded by the line through p parallel to $a_{i+2}a_{i+3}$ such that $a_i \notin H_i$. Observe that if p' is in H_i, then $d_P(a_i, p) < d_P(a_i, p')$. Let H be $H_1 \cap H_2 \cap H_3$. Notice that $H' = H \cap \mathrm{int}Q$ is a nonempty open set, and that p is a boundary point of H'. If p' is a point of H', then $d_P(a_i, p') > d_P(a_i, p) \geq \lambda(P, p)$ for $i = 1, 2, 3$. Moreover, if p' is close enough to p, then $d_P(a_j, p') > \lambda(P, p)$ for $j = 4, 5$. Thus, we can choose a point $p' \in \mathrm{int}Q$ such that $\lambda(P, p) < \lambda(P, p')$. Hence $\lambda(P, p)$ cannot be maximal.

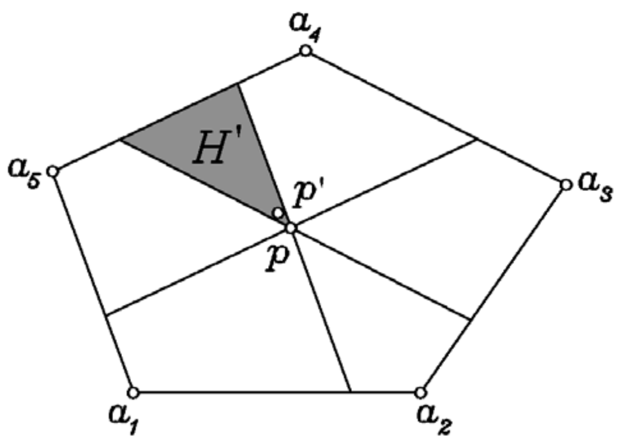

Figure 1

Subcase 2.2, when P has exactly two, nonconsecutive vertices in P-distance from p greater than $\lambda(P, p)$. Without loss of generality, let $d_P(a_1, p) = d_P(a_2, p) = d_P(a_4, p) = \lambda(P, p)$, $d_P(a_3, p) > \lambda(P, p)$ and $d_P(a_5, p) > \lambda(P, p)$. Take the convex pentagon $P' = a_1a_2a_3a_4a_5'$, where a_5' is an interior point of the segment a_5a_1. We have $d_{P'}(a_2, p) > d_P(a_2, p) = \lambda(P, p)$ and $d_{P'}(a_5', p) < d_P(a_5, p)$. Moreover, if a_5' is close enough to a_5, then $d_{P'}(a_5', p) > \lambda(P, p)$, and p is in the kernel of P'. Hence, according to Subcase 2.1, there exists a point p' in the kernel of P' such that $\lambda(P, p) = \lambda(P', p) < \lambda(P', p')$.

Subcase 2.3, when P has exactly one vertex in P-distance from p greater than $\lambda(P, p)$. Let this vertex be a_5. Take the convex pentagon $P^* = a_1a_2a_3a_4a_5^*$, where a_5^* is an interior point of the segment a_5a_1. We have that $d_{P^*}(a_2, p) > d_P(a_2, p) = \lambda(P, p)$ and $d_{P^*}(a_5^*, p) < d_P(a_5, p)$. Moreover, if a_5^* is close enough to a_5, then $d_{P^*}(a_5^*, p) > \lambda(P, p)$. Hence, thanks to Subcase 2.2, there exist a convex pentagon P' and a point p' in the kernel of P' such that $\lambda(P, p) = \lambda(P^*, p) < \lambda(P', p')$.

Subcase 2.4, when $d_P(a_i, p) = \lambda(P, p)$ for $i = 1, \dots, 5$. As we are looking for the maximal value of $\lambda(P, p)$, we assume that $\lambda(P, p) > 1$. For the sake of simplicity, we use the notation $\nu(P, p) = \frac{\lambda(P,p)}{2-\lambda(P,p)}$. Thus $\frac{|a_i p|}{|p b_i|} = \nu(P, p)$ for $i = 1, \dots, 5$. Observe that $\nu(P, p)$ is a strictly increasing function of $\lambda(P, p)$. Additionally, $\lambda(P, p) > 1$ implies that $\nu(P, p) > 1$. Let h_p be the homothety with homothety center p and with homothety ratio

$-\frac{1}{\nu(P,p)}$. Then $h_p(a_i) = b_i$ for $i = 1, \ldots, 5$.

Consider the intersection point a of the lines $L(a_1, a_5)$ and $L(a_2, a_3)$. Let us take a Cartesian coordinate system. As the relative distance of two points does not change under affine transformations, we can assume that the points a, a_1, a_2 are $(0,0)$, $(1,-1)$, $(-1,-1)$, respectively (see Figure 2).

We intend to show that if $p^x \neq 0$, then $\lambda(P,p)$ is not maximal. Assume that $p^x > 0$ (in the other case the proof is analogous).

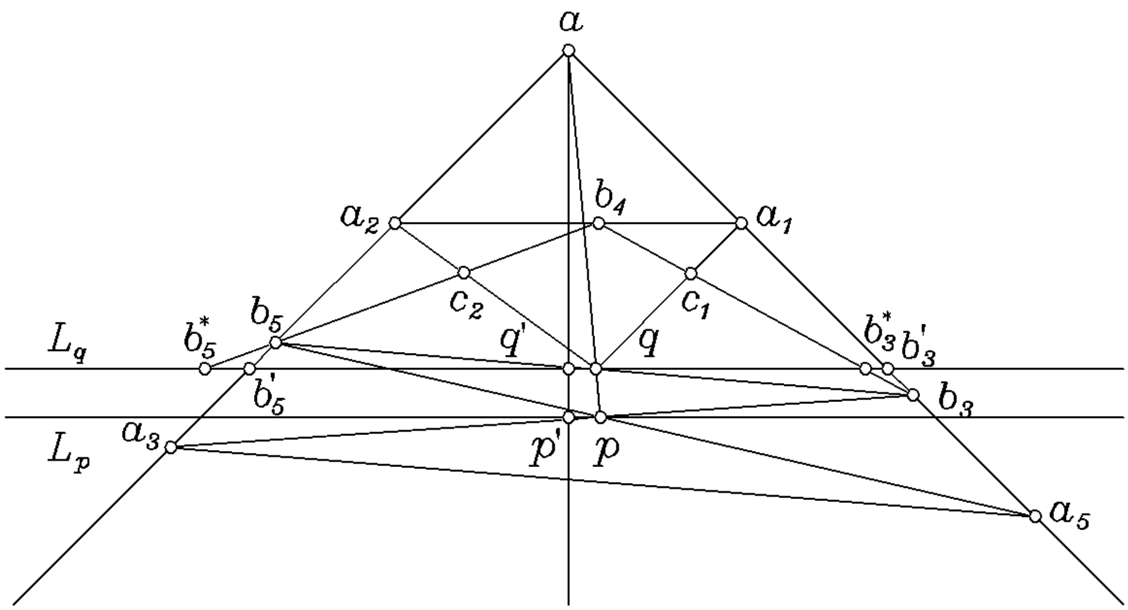

Figure 2

Take the intersection point q of the segments ap and b_3b_5. Denote the straight lines $y = p^y$ and $y = q^y$ by L_p and L_q, respectively. Let p' and q' be the points $(0, p^y)$ and $(0, q^y)$, respectively, and let $h_{p'}$ be the homothety with center p' and with homothety ratio $-\frac{1}{\nu(P,p)}$. Let us denote by b_3' and by b_5' the intersections of L_q and of the straight lines $L(a, a_1)$ and $L(a, a_2)$, respectively. Let a_3' be the intersection of $L(a, a_2)$ and $L(p', b_3')$. Similarly, let a_5' be the intersection of $L(a, a_1)$ and $L(p', b_5')$.

We show that $b_3' = h_{p'}(a_3')$ and that $b_5' = h_{p'}(a_5')$. Observe that $pb_3b_5 = h_p(pa_3a_5)$. Thus a_3a_5 and b_3b_5 are parallel and b_3b_5 is the homothetic image of a_3a_5 of ratio $\frac{1}{\nu(P,p)}$, where the center of homothety is a. Since $a_3'a_5'$ and $b_3'b_5'$ are also parallel, $b_3'b_5'$ is the homothetic image of $a_3'a_5'$ of ratio $\frac{1}{\nu(P,p)}$, where the center of homothety is a. Hence $\frac{|b_3'b_5'|}{|a_3'a_5'|} = \frac{1}{\nu(P,p)}$. From this we get that $b_3'b_5'p' = h_{p'}(a_3'a_5'p')$. That is, $b_3' = h_{p'}(a_3')$ and $b_5' = h_{p'}(a_5')$.

Denote a_1 by a_1', a_2 by a_2', $h_{p'}(a_1)$ by b_1' and $h_{p'}(a_2)$ by b_2'. Let a_4' be the common point of the straight lines $L(a_3', b_1')$ and $L(a_5', b_2')$ and let b_4' denote $h_{p'}(a_4')$. Using these notations we have $\frac{2|a_i'p'|}{|a_i'b_i'|} = \lambda(P,p)$ for $i = 1, \ldots, 5$. We omit a consideration which shows that $P' = a_1'a_2'a_3'a_4'a_5'$ is a convex pentagon, that p' is in the kernel of P' and that P' has no side of P'-length 2. From the above properties of P' and p' we get that $d_{P'}(p', a_i') = \lambda(P, p)$ for $i = 1, 2, 3, 5$. We show that $d_{P'}(p', a_4') > \lambda(P, p)$.

Take the points $c_1 = h_p(b_1)$, $c_2 = h_p(b_2)$, $c_1' = h_{p'}(b_1')$ and $c_2' = h_{p'}(b_2')$. As the homothety ratios of h_p and $h_{p'}$ are equal, we have $c_1^y = c_2^y = {c_1'}^y = {c_2'}^y$ and $|c_1c_2| = |c_1'c_2'|$. Since b_3b_5 and a_3a_5 are parallel, the quadrangle $a_5b_3b_5a_3$ is a trapezoid. Thus $|b_3q| = |qb_5|$. Consider the triangles $b_3b_3'q$ and $b_5b_5'q$. We get that $|b_3b_3'| = |b_5b_5'|$. Let b_3^* be the intersection point of the segment b_3b_4 and the straight line L_q. Similarly, let b_5^* be the intersection point of the segment b_4b_5 and the straight line L_q. Notice that $b_4^x > 0$, $b_3 \in b_3'a_5$ and that $b_5 \in b_5'a_2$. These observations and the fact that $|b_3b_3'| = |b_5b_5'|$ imply that $|b_3^*b_5^*| > |b_3'b_5'|$. Consider that b_4 is the intersection of $L(b_3^*, c_1)$ and $L(b_5^*, c_2)$ and that b_4' is the intersection of $L(b_3', c_1')$ and $L(b_5', c_2')$. As $|b_3^*b_5^*| > |b_3'b_5'|$ and $|c_1c_2| = |c_1'c_2'|$, we get that $b_4^y < {b_4'}^y$. Take the intersection point b_4^* of a_1a_2 and $p'b_4'$. Since $\frac{|p'a_4'|}{|p'b_4^*|} > \nu(P, p)$, we have $d_{P'}(p', a_4') > \lambda(P, p)$. Obviously, $\lambda(P, p) = \lambda(P', p')$. Thus, according to Subcase 2.3, the value $\lambda(P, p)$ cannot be maximal.

Notice that our choice of the side a_1a_2 was arbitrary. This implies that $\lambda(P, p)$ can be maximal only if P is affine symmetric to every line containing the midpoint of a side of P and the opposite vertex of P and if p is on every one of the above lines. But this holds only if P is an affine regular pentagon and if p is its center.

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Lemma 2. *Let $P = a_1a_2a_3a_4a_5$ be a convex pentagon and let p be a point of P which is not in the kernel of P. Then among $p, a_1, \ldots, a_5$ there exists a pair of points in P-distance less than λ.*

Proof. If P is a degenerate pentagon, then it has a chord containing at least 3 vertices of P. Thus in this case P has a side of P-length at most 1, which is less than λ.

In the sequel we deal with the case when P is nondegenerate. Take a Cartesian coordinate system. As the P-distance of two points is affine invariant, we assume that the points a_1, a_2 and a_5 are $(0, 0)$, $(1, 0)$ and $(0, 1)$, respectively. Let b be the point $(1, 1)$. Denote the square $a_1a_2ba_5$ by S. Furthermore, for every $i, j \in \{1, \ldots, 5\}$ where $i \neq j$, we denote the slope of the line $L(a_i, a_j)$ by m_{ij}, provided it exists.

Case 1, when P has more than one side of P-length 2. Consider the case when P has two nonconsecutive sides of P-length 2. We assume that $d_P(a_1, a_2) = 2$ and that $d_P(a_3, a_4) = 2$ (the proof of the other cases is analogous). Let the angle of P at the vertex a_i be denoted by α_i for $i = 1, \ldots, 5$. From $d_P(a_1, a_2) = 2$ we get that $\alpha_1 + \alpha_2 \leq \pi$. Similarly, from $d_P(a_1, a_2) = 2$ we have $\alpha_3 + \alpha_4 \leq \pi$. The convexity of P implies that $\alpha_5 \leq \pi$. Obviously, $\sum_{i=1}^{5} \alpha_i = 3\pi$. Thus $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = \alpha_5 = \pi$. Therefore P is a degenerate pentagon.

Let us assume that P has two consecutive sides of P-length 2. Without loss of generality, let these sides be a_5a_1 and a_1a_2. Hence $P \subset S$. Denote the triangle $a_1a_2a_5$ by

S_1. Take the homothetical copies S_2, S_3, S_4 of a_2ba_5, S, a_2ba_5 with ratio $\frac{1}{2}$ and with homothety centers a_2, b, a_5, respectively. As P is convex, S_1 contains neither of the points a_3 and a_4 in its interior. If S_2 contains a_3 or if S_4 contains a_4, then $d_P(a_2, a_3) \leq 1$ or $d_P(a_4, a_5) \leq 1$, respectively. Finally, if S_3 contains both a_3 and a_4, then $m_{34} < 0$ implies that $d_P(a_3, a_4) \leq 1$. Thus we get that P has a side of P-length at most 1.

Case 2, when P has exactly one side of P-length 2. We choose the indices of the points such that $d_P(a_1, a_2) = 2$ and that $a_3^y \geq 1$. The condition of this case and the convexity of P imply that $0 < a_4^x < a_3^x \leq 1$ and that $1 \leq a_3^y < a_4^y$. Moreover, either $d_P(a_1, a_5) < \lambda$ or $a_4^y \leq \frac{1}{\tau}$. In the following we assume that $a_4^y \leq \frac{1}{\tau} < 2$.

Observe that for arbitrary $w \in E^2$, the set of points whose P-distance from w is less than λ is the interior of the translate of $\frac{\tau}{2}(P - P)$ where the center of the body is w. From the previous considerations concerning the properties of P we get that the sides of the centrally symmetric convex decagon $\frac{\tau}{2}(P - P)$ are parallel to a_1a_5, a_4a_5, a_1a_2, a_3a_4, a_2a_3.

First we show that if every side of P has P-length greater than 1, then $m_{45} > m_{13}$. We show the statement indirectly. Denote the intersection point of $L(a_2, a_3)$ and $L(a_4, a_5)$ by s and denote the intersection point of a_1a_3 and a_2a_5 by q. Let a_3' and a_4' be the homothetic images of a_3 and a_4, respectively, where the center of homothety is s and its ratio is 2. Denote the midpoint of the segment a_5s by s_5. Let t be the point of a_2a_3 such that a_1t and a_4a_5 are parallel. Similarly, let s' be the point of $L(a_2, a_3)$ such that a_5s' and a_1a_3 are parallel. Observe that a_1t is a maximal chord of P parallel to a_4a_5. Thus $d_P(a_4, a_5) > 1$ implies that $\frac{1}{2}|a_1t| < |a_4a_5|$. Moreover, we have $|a_5s_5| = \frac{1}{2}|a_5s| \leq \frac{1}{2}|a_1t|$. Therefore $|a_5s_5| < |a_4a_5|$, from which $a_4 \in s_5s$. Since $m_{13} \geq 1$, we get that $1 \leq \frac{|a_2q|}{|qa_5|} = \frac{|a_2a_3|}{|a_3s'|}$. Hence $\frac{|a_2a_3|}{|a_3s|} \geq \frac{|a_2a_3|}{|a_3s'|} \geq 1$. This and $a_4 \in s_5s$ imply that $a_3'a_4'$ is a chord of C. Thus $d_P(a_3, a_4) \leq 1$, which is a contradiction.

Now we prove the statement under the condition that $m_{45} > m_{13}$. If p is in both the triangles $a_1a_3a_5$ and $a_2a_3a_5$, then, according to the proof of Case 1 in Lemma 1 we have $d_P(a_3, p) \leq 1$ or $d_P(a_5, p) \leq 1$. We intend to examine the cases when $p \in a_3a_4a_5$, $p \in a_1a_2a_5$ or $p \in a_2a_3q$.

Subcase 2.1, when p is in the triangle $a_3a_4a_5$. Denote the midpoints of the segments a_4a_5, a_3a_5 and a_3a_4 by c_3, c_4 and c_5, respectively. Notice that the triangles $a_3c_4c_5$, $a_5c_3c_4$ and the parallelogram $a_4c_5c_4c_3$ are contained in the homothetical copies of P with homothety ratio $\frac{1}{2}$ where the homothety centers are a_3, a_5 and a_4, respectively. Thus in this case at least one of the values $d_P(a_3, p)$, $d_P(a_4, p)$, $d_P(a_5, p)$ is at most 1.

Subcase 2.2, when p is in the triangle $a_1a_2a_5$. We show the statement indirectly, therefore we assume that among p and the vertices of P there is no pair in P-distance less than λ. Denote by Q_1 and by Q_5 the translates of $\frac{\tau}{2}(P-P)$ where the centers of the bodies are a_1 and a_5, respectively. Consider the points $b_1 = (1-\tau, 0)$ and $b_5 = (1-\tau, 1-\tau)$. As $d_P(a_2, p) \geq \lambda$, we have $p \in a_1b_1b_5a_5$. We show that $a_1b_1b_5a_5$ is covered by the interiors of Q_1 and Q_5. For this it is enough to show that b_1b_5 is in the interior of $Q_1 \cup Q_5$. Denote by d_1 the intersection of b_1b_5 and of the boundary of Q_1 such that $d_1 \neq b_1$. Similarly, let d_5

be the intersection of b_1b_5 and of the boundary of Q_5. We omit an easy calculation that if d_1 is not on the side of Q_1 parallel to a_2a_3, or if d_5 is not on the side of Q_5 parallel to a_4a_5, then b_1b_5 is in the interior of $Q_1 \cup Q_5$. In the opposite case we get that $d_1^y = m_{23}(1-2\tau)$, and that $d_5^y = m_{45}(1-2\tau)+1-\tau$. Thus $d_1^y - d_5^y = (2\tau-1)(m_{45}-m_{23})+\tau-1$.

Let us assume that $m_{34} \le -1$. Take the point u on the line $L(a_2,a_3)$ such that a_3 is the midpoint of the segment a_2u. As $d_P(a_3,a_4) \ge \lambda > 1$, we have $\angle ua_5b < \angle a_4a_5b$. Thus, $0 < \frac{2a_3^y-1}{2a_3^x-1} < m_{45}$. This implies that $d_1^y - d_5^y \ge (2\tau-1)(\frac{2a_3^y-1}{2a_3^x-1}+\frac{a_3^y}{1-a_3^x})+\tau-1 \ge (2\tau-1)(\frac{1}{2a_3^x-1}+\frac{1}{1-a_3^x})+\tau-1$. But the last expression is always positive.

Now we discuss the case when $m_{34} > -1$. In this case from $d_P(a_3,a_4) \ge \lambda$ we conclude that $a_3^x - a_4^x \ge \tau$. Consider the point $m = (0, \frac{1}{1-\tau})$. Take the line L_m through m with slope -1. Since $m_{34} > -1$ and since $d_P(a_3,a_4) \ge \lambda$, a_3 and a_4 are in the open halfplane not containing a_1 bounded by L_m. As $a_4^y \le \frac{1}{\tau}$, we have $a_4^x > \frac{1}{1-\tau} - \frac{1}{\tau}$ and thus $a_3^x > \frac{1}{1-\tau} - \frac{1}{\tau} + \tau = \frac{11\sqrt{5}-5}{20} \approx 0.980$. But this contradicts that d_1 is on the side of Q_1 parallel to a_2a_3, that is, that $a_3^x \le \frac{1-\tau}{\tau} = \frac{\sqrt{5}+1}{4} \approx 0.809$. Hence b_1b_5 is in the interior of $Q_1 \cup Q_5$. Therefore every point of $a_1b_1b_5a_5$ is in P-distance from a_1 or from a_5 less than λ.

Subcase 2.3, when p is in the triangle a_2a_3q. Let Q_2 and Q_3 be the translates of $\frac{\tau}{2}(P-P)$ where the centers of the bodies are a_2 and a_3, respectively. If $p^y > 1$, then p is in the interior of Q_3. In the following we deal with the case when $p^y \le 1$. From $d_P(a_5,p) \ge \lambda$ we have $p^x \ge \tau$. We show that the points of a_2a_3q with x-coordinates at least τ are in the interior of $Q_2 \cup Q_3$. Denote by e_2 the common point of the line $x=\tau$ and of the boundary of Q_2 with greater y-coordinate. Denote by e_3 the common point of the line $x=\tau$ and of the boundary of Q_3 with less y-coordinate. Let us show that $e_2^y - e_3^y$ is positive. We have $e_3^y \le (1-\tau)a_3^y$. Moreover, $e_2^y = \tau a_4^y$ or $e_2^y = m_{45}(2\tau-1)+\tau$. If $e_2^y = \tau a_4^y$, then $e_2^y - e_3^y \ge \tau(a_3^y + a_4^y) - a_3^y > 0$. In the sequel we assume that $e_2^y = m_{45}(2\tau-1)+\tau$. Observe that $m_{45} \ge m_{13} \ge 1$. Hence, if $a_3^y < \frac{3\tau-1}{1-\tau}$, then $e_2^y - e_3^y \ge 3\tau-1-(1-\tau)a_3^y > 0$. Let us assume the opposite case, when $a_3^y \ge \frac{3\tau-1}{1-\tau}$. In this case $a_4^y \le \frac{1}{\tau} < a_3^y + \tau$. Thus $d_P(a_3,a_4) \ge \lambda$ implies that $a_4^x \le a_3^x - \tau \le 1-\tau$. Take the points $m(0,\frac{1}{1-\tau})$ and $g(\frac{1}{\tau}, 1-\tau)$. Denote by h the intersection point of $L(m,g)$ and $x=1$. We omit an elementary calculation which shows that $h^y = \frac{1}{\tau} - \frac{2\tau-1}{(1-\tau)^2}$. Since $d_P(a_3,a_4) \ge \lambda$, we get that a_3 is in the closed halfplane containing a_1 bounded by $L(a_4,m)$. Therefore $a_4^y - a_3^y \ge \frac{1}{\tau} - h^y$. Thus, $a_3^y \le a_4^y - \frac{1}{\tau} + h^y \le h^y \approx 1.281$. But this contradicts our assumption that $a_3^y \ge \frac{3\tau-1}{1-\tau} \approx 1.472$.

We have shown that $e_2^y - e_3^y$ is positive. But this implies that every point of the triangle a_2a_3q with x-coordinate at least τ is in the interior of $Q_2 \cup Q_3$.

Case 3, when P has no side of P-length 2. We assume that p is in the triangle $a_1a_2a_5$ and that m_{34} is at least -1 (the proof of the other cases is analogous). Since P has no side of P-length 2, we have $a_3^x > 1$ and $a_4^y > 1$. Observe that the points of $a_1a_2a_5$ with x-coordinate greater than $1-\tau$ are in P-distance from a_2 less than λ. Similarly, the points of $a_1a_2a_5$ with y-coordinate greater than $1-\tau$ are in P-distance from a_5 less than λ. Thus it is enough to deal with the case when both coordinates of p are at most $1-\tau$. Take the point $f=(\frac{1}{\tau}-1,\frac{1}{\tau}-1)$. We intend to show that if P has no side of P-length less than λ, then f is in the interior of P.

Consider the case when the maximal chord parallel to a_4a_5 has an endpoint at a_1. In this case the other endpoint of the above chord is on the segment a_3a_4. This and $d_P(a_4,a_5)\geq\lambda$ imply that the y-coordinate of the common point of $L(a_3,a_4)$ and of the line $y=0$ is at least $\frac{1}{1-\tau}=\sqrt{5}$. Therefore, as $-1\leq m_{34}$, we get that f is in the open halfplane containing a_1 bounded by $L(a_3,a_4)$. Thus f is in the interior of P.

Consider the case when the maximal chord of P parallel to a_4a_5 has an endpoint at a_3. In this case $d_P(a_4,a_5)\geq\lambda$ implies that $a_4^x\geq\tau a_3^x$. Since $d_P(a_3,a_4)\geq\lambda$, we have $a_3^x-a_4^x\geq\tau$. Therefore $a_4^x\geq\frac{\tau^2}{1-\tau}$. From $d_P(a_2,a_3)\geq\lambda$ we get that a_3 is not in the interior of the homothetical copy of $a_1a_2a_5$ with homothety ratio τ where the image of a_1 is a_2. Take the points $a_4'=(\frac{\tau^2}{1-\tau},1)$ and $a_3'=(1,1-\tau)$. We omit an elementary calculation which shows that f is in the open halfplane containing a_1 bounded by $L(a_3',a_4')$. Thus f is in the open halfplane containing a_1 bounded by $L(a_3,a_4)$. Therefore f is in the interior of P.

We have shown that if P has no side of P-length less than λ, then f is in the interior of P. But the definition of f and our inequalities for the coordinates of p imply that in this case $d_P(p,a_1)<\lambda$.

∎

Lemma 3. *Let $a_1,\ldots,a_6$ be points such that their convex hull Q is a quadrangle or a triangle. Then among those points there exists a pair in Q-distance at most* 1.

Proof. We show the statement of our lemma indirectly, we assume that among the points $a_1,\ldots,a_6$ there is no pair in Q-distance at most 1. Let us take a Cartesian coordinate system. As the Q-distance of two points does not change under affine transformation, we assume that the points a_1, a_2 and a_3 are $(0,1)$, $(0,0)$ and $(1,0)$, respectively. Take the point $b(1,1)$ and the square $S=a_1a_2a_3b$. We choose the indices of our points such that $Q\subset S$. Let us denote the homothetical copies of S with homothety ratio $\frac{1}{2}$ and with centers a_1, a_2, a_3, b by S_1, S_2, S_3, S_4, respectively. Consider the center c of S, the center b_1 of the segment a_1a_2 and the center b_2 of the segment a_2a_3. Observe that every point of the triangle a_1b_1c is in Q-distance at most 1 from a_1. Similarly, every point of the triangles $a_2b_2b_1$ and b_2a_3c is in Q-distance at most 1 from a_2 and from a_3, respectively. Notice that there are no two points in the triangle b_1b_2c in Q-distance from each other greater than 1. Thus b_1b_2c contains at most one of the points a_4, a_5, a_6. Hence Q is a quadrangle. Let a_4 be the fourth vertex of Q. As $d_Q(a_1,a_4)>1$ and $d_Q(a_3,a_4)>1$, we have $a_4\in S_4$. Hence every point of S_1 and S_3 is in Q-distance at most 1 from a_1 and a_3, respectively. Furthermore, $S_4\cap Q$ is covered by the homothetical copy of Q with homothety center a_4

and with ratio $\frac{1}{2}$. Thus, every point of $S_4 \cap Q$ is in Q-distance at most 1 from a_4. Moreover, b_2cb_1 contains at most one of the points a_5 and a_6, which is a contradiction.

■

Proof of Theorem. First, observe that if C is an affine regular pentagon, and if the six points are its vertices and its center, then the minimal pairwise C-distance of the points is λ.

Take an arbitrary plane convex body C. Let $p_1, \ldots, p_6$ be points of C. Let us denote the convex hull of $p_1, \ldots, p_6$ by C'. As $C' \subset C$, the C'-distance of arbitrary two points is greater than or equal to their C-distance. Theorem in [3] says that every convex hexagon has a side of relative length at most $8-4\sqrt{3} \approx 1.072$. Thus, if C' is a hexagon, then among $p_1, \ldots, p_6$ there is a pair in C'-distance at most $8-4\sqrt{3}$, which is less than λ. With respect to Lemma 1 and Lemma 2, if C' is a pentagon, then the minimal pairwise C'-distance of the points is at most λ, with equality if and only if C' is an affine regular pentagon and the points are its vertices and its center. According to Lemma 3, if C' is a quadrangle or a triangle, then there exists a pair of points in C'-distance at most 1, which is less than λ. We have proved the first statement of our theorem.

To prove the second statement, it remains to show that if C' is an affine regular pentagon, the points are its vertices and its center and if there is no pair of them in a C-distance less than λ, then $C = C'$. Let us choose the indices of the points such that C' is the pentagon $p_1p_2p_3p_4p_5$ and that p_6 is the center of C'. Assume that $C \neq C'$. In this case there exists a point $q \in C$, which is not a point of C' and the convex hull D of q and C' is a convex hexagon. It is enough to deal with the case when $D = p_1p_2p_3p_4p_5q$ (the proof of the other cases is analogous). But then $d_C(p_6, p_3) \leq d_D(p_6, p_3) < d_{C'}(p_6, p_3) = \lambda$.

■

Finally, we conjecture that among seven arbitrary points of an arbitrary plane convex body there is a pair in relative distance at most 1. The value 1 is attained, for instance, for the parallelogram. The example of the parallelogram shows that the estimate 1 is attained even for eight or nine points.

Corollary of [3] says that every convex heptagon has a side of relative length at most 1. From this and from Lemma 3 we immediately get that among eight or nine arbitrary points of an arbitrary plane convex body there is a pair in relative distance at most 1.

According to [2] or [4], we also conjecture that no plane convex body can be packed by its seven homothetical copies of ratio greater than $\frac{1}{3}$. Analogously, we observe that no plane convex body can be packed by its eight or nine homthetical copies of ratio greater than $\frac{1}{3}$.

Appendix

Statement. *Among five arbitrary points of an arbitrary plane convex body there exists a pair in relative distance at most* $\sqrt{5}-1$.

Proof. Let C be a plane convex body. Let $a_1, \ldots, a_5$ be points of C. Let us denote by C' the convex hull of $a_1, \ldots, a_5$. Observe that the C'-distance of any two points is greater than or equal to their C-distance. Thus it is enough to show that among $a_1, \ldots, a_5$ there exists a pair in C'-distance at most $\sqrt{5}-1$.

First, let us assume that C' is a triangle. Let us take the three segments whose endpoints are the midpoints of the sides of C'. These segments divide C' into four homothetical copies of C', where the homothety ratio of three of them is $\frac{1}{2}$ and the homothety ratio of the remaining one is $-\frac{1}{2}$. Observe that none of the above homothetical copies contains two points in C'-distance greater than 1. Moreover, there is a copy containing at least two of the points $a_1, \ldots, a_5$. Hence, among the points $a_1, \ldots, a_5$ there is a pair in C'-distance at most 1, which is less than $\sqrt{5}-1$.

Second, let us assume that C' is a quadrangle. Observe that there exists a parallelogram P containing C' such that two its consecutive sides are sides of C'. Without loss of generality, let $C' = a_1a_2a_3a_4$ and let a_1, a_2, a_3 be vertices of P. Denote the fourth vertex of P by b.

If a_5 is a boundary point of C', then the C'-distance of a_5 and a vertex of C' is at most 1. Thus it is enough to deal with the case when a_5 is in the interior of C'.

In the following, let us assume indirectly that the pairwise C'-distances of a_1, a_2, a_3, a_4 and a_5 are greater than $\sqrt{5}-1$. Let P_1 and P_3 be the homothetic images of P with homothety ratio $\frac{1}{2}(\sqrt{5}-1)$, where the centers of homothety are a_1 and a_3, respectively. Let P_2 and P_4 be the homothetic images of P with homothety ratio $\frac{1}{2}(3-\sqrt{5})$, where the centers of homothety are a_2 and b, respectively. Hence $P = P_1 \cup P_2 \cup P_3 \cup P_4$.

Notice that the C'-distance of every point of P_1 and P_3 is at most $\sqrt{5}-1$ from a_1 and from a_3, respectively. Hence a_4 and a_5 are in P_2 or P_4. As C' is convex, $a_4 \notin P_2$ and therefore it is in P_4. Observe that the C'-distance of every point of $P_4 \cap Q$ is less than $\sqrt{5}-1$ from a_4. Thus a_5 is in P_1.

Let $u = P_1 \cap P_3 \cap P_4$ and let $v = P_1 \cap P_2 \cap P_3$. Notice that v is the homothetic image of u, where the center of homothety is a_1 and the homothety ratio is $\frac{1}{2}(\sqrt{5}-1)$. Therefore P_1 is contained in the homothetic image of C', where the center of the homothety is a_1 and the homothety ratio is $\frac{1}{2}(\sqrt{5}-1)$. From this we have $d_{C'}(a_1, a_5) \leq \sqrt{5}-1$, which contradicts our indirect assumption.

The proof of the third case, when C' is a convex pentagon, is given in a detailed form in [1].

■

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Károly Böröczky
Eötvös Loránd University, Department of Geometry
1117 Budapest, Pázmány P. sétány 1/c, Hungary
E-mail: boroczky@cs.elte.hu

Zsolt Lángi
College of Dunaújváros, Institute of Natural Sciences
2400 Dunaújváros, Táncsics M. út 1/a, Hungary
E-mail: zslangi@kac.poliod.hu