A1 Vizsga, 2022.12.16. m.o.

A) Az $\{a_n\}$ sorozat konvergens és határértéke $A \in \mathbb{R}$, ha minden $\varepsilon > 0$ esetén van olyan $n_\varepsilon \in \mathbb{R}$, hogy ha $n > n_\varepsilon$, akkor $|a_n - A| < \varepsilon$.

B) Legyen $f: [a,b] \to \mathbb{R}$ folytonos $[a,b]$-n. Ekkor f minden $f(a)$ és $f(b)$ közé eső értéket felvesz.

C) Az $\underline{a}, \underline{b}$ térvektorok vektoriális szorzata az az $\underline{a} \times \underline{b}$ vektor, melyre
- $|\underline{a} \times \underline{b}| = |\underline{a}| \cdot |\underline{b}| \cdot \sin\gamma$, ahol γ az $\underline{a}, \underline{b}$ által bezárt szög,
- $\underline{a} \times \underline{b}$ merőleges $\underline{a}$-ra és $\underline{b}$-re,
- $\underline{a}, \underline{b}, \underline{a} \times \underline{b}$ ebben a sorrendben jobbsodrású rendszert alkot.

(1) Legyen $z = x + yi$, $x, y \in \mathbb{R}$. Ekkor $\operatorname{Im}((x+yi)^2) = \operatorname{Im}((x-yi)^2) \Leftrightarrow \operatorname{Im}(x^2 + 2xyi - y^2) = \operatorname{Im}(x^2 - 2xyi - y^2)$ $\Leftrightarrow 2xy = -2xy \Leftrightarrow xy = 0 \Leftrightarrow x = 0$ vagy $y = 0$. A megoldás a 2 koordinátatengely uniója:

(2) $n = 1$ esetén $\frac{1}{2^1} = \frac{1}{2} = 2 - \frac{3}{2^1}$

Tfh. $n = m$ esetén igaz az állítás. Belátjuk $n = m+1$ esetén:

$$\frac{1}{2^1} + \frac{2}{2^2} + \dots + \frac{m+1}{2^{m+1}} = \left(\frac{1}{2^1} + \dots + \frac{m}{2^m}\right) + \frac{m+1}{2^{m+1}} = 2 - \frac{m+2}{2^m} + \frac{m+1}{2^{m+1}} = 2 - \frac{2(m+2) - (m+1)}{2^{m+1}} =$$

$= 2 - \frac{m+3}{2^{m+1}}$, és ez akartuk igazolni.

(3) $f(x) = x^2 \ln x$, $D_f = (0; \infty)$, zérushely: $x = 1$. Nem páros, nem páratlan, nem per.

Folytonos D_f-en. $\lim\limits_{x \to \infty} x^2 \ln x = \infty$

$$\lim_{x \to 0+0} x^2 \ln x = \lim_{x \to 0+0} \frac{\ln x}{1/x^2} \overset{\frac{-\infty}{\infty}}{=} \lim_{x \to 0+0} \frac{1/x}{(-2) \cdot x^{-3}} = \lim_{x \to 0+0} -\frac{1}{2}x^2 = 0$$

$f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = x(2\ln x + 1)$ $\qquad f'(x) = 0 \Leftrightarrow \ln x = -\frac{1}{2}$ ($x > 0$ miatt) $\Leftrightarrow x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

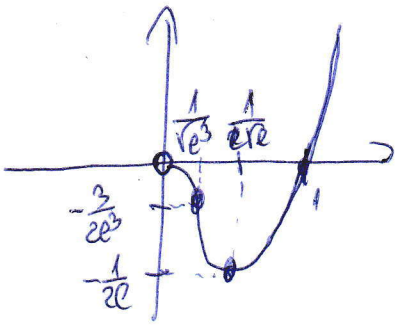
	$0 < x < \frac{1}{\sqrt{e}}$	$x = \frac{1}{\sqrt{e}}$	$\frac{1}{\sqrt{e}} < x$
f'	$-$	0	$+$
f	$\downarrow$	lok. min.	$\uparrow$

$f\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2e}$

$f''(x) = 1 \cdot (2\ln x + 1) + x \cdot 2 \cdot \frac{1}{x} = 2\ln x + 3$ $\qquad f''(x) = 0 \Leftrightarrow \ln x = -\frac{3}{2} \Leftrightarrow x = e^{-\frac{3}{2}} = \frac{1}{\sqrt{e^3}}$

	$0 < x < \frac{1}{\sqrt{e^3}}$	$x = \frac{1}{\sqrt{e^3}}$	$\frac{1}{\sqrt{e^3}} < x$
f''	$-$	0	$+$
f	$\cap$	infl. pont	$\cup$

$f\left(\frac{1}{\sqrt{e^3}}\right) = -\frac{3}{2e^3}$



$R_f = \left[-\frac{1}{2e}; \infty\right)$

(4)
$$\lim_{x\to 1+0} (x-1)\cdot tg\left(\frac{\pi}{2}x\right) = \lim_{x\to 1+0} \frac{x-1}{ctg\left(\frac{\pi}{2}x\right)} = \lim_{x\to 1+0} \frac{1}{\frac{-1}{\sin^2\left(\frac{\pi}{2}x\right)}\cdot\frac{\pi}{2}} = -\frac{2}{\pi}$$

$\frac{0}{0}$

(5)
$$\int \frac{3x-2}{x^2-6x+13}\,dx = \int \frac{\frac{3}{2}(2x-6)+7}{x^2-6x+13}\,dx = \frac{3}{2}\int \frac{2x-6}{x^2-6x+13}\,dx + \int \frac{7}{(x-3)^2+4}\,dx = \frac{3}{2}\ln\left|x^2-6x+13\right| +$$

$(x^2-6x+13)' = 2x-6$

$$+\frac{7}{4}\cdot\int \frac{1}{\left(\frac{x-3}{2}\right)^2+1}\,dx = \frac{3}{2}\ln\left|x^2-6x+13\right| + \frac{7}{4}\cdot\frac{arctg\,\frac{x-3}{2}}{1/2} + C$$

(6)
$$\int x^2 e^{-x}\,dx = x^2(-e^{-x}) - \int 2x\cdot(-e^{-x})\,dx = -x^2e^{-x} + \int 2x\,e^{-x}\,dx = -x^2e^{-x} + 2x(-e^{-x}) - \int 2(-e^{-x})\,dx =$$

$f = x^2 \quad f' = 2x$
$g' = e^{-x} \quad g = \frac{e^{-x}}{-1} = -e^{-x}$

$f = 2x \quad f' = 2$
$g' = e^{-x} \quad g = -e^{-x}$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

(7)
$$\int_0^9 \frac{3}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{3}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{3}{\sqrt[3]{x-1}}\,dx = \lim_{a\to 1-0}\int_0^a \frac{3}{\sqrt[3]{x-1}}\,dx + \lim_{b\to 1+0}\int_b^9 \frac{3}{\sqrt[3]{x-1}}\,dx = \lim_{a\to 1-0}\left[3\cdot\frac{(x-1)^{2/3}}{\frac{2}{3}}\right]_0^a +$$

$$+\lim_{b\to 1+0}\left[\frac{9}{2}(x-1)^{2/3}\right]_b^9 = \lim_{a\to 1-0}\left(\frac{9}{2}(a-1)^{2/3} - \frac{9}{2}\right) + \lim_{b\to 1+0}\left(\frac{9}{2}\cdot 8^{2/3} - \frac{9}{2}(b-1)^{2/3}\right) = -\frac{9}{2} + \frac{9}{2}\cdot 4 = \frac{27}{2}$$