Convex Geometry

1st midterm - SAMPLE

1) Consider the polynomial $p(t) = \sum_{i=0}^{n} a_i t^i$ as a point $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$. Let

$$V = \{p \in \mathbb{R}^{n+1} : p(t) \geq 0 \text{ for every } t \in \mathbb{R}\}.$$

Prove that V is a closed, convex set.
(5 points)

Solution: Let $p, q \in V$, and let $\lambda \in [0,1]$. We need to show that the polynomial $h = \lambda p + (1-\lambda)q$ is in V. Since $h(t) = \lambda p(t) + (1-\lambda)q(t) \geq t$ for all $t \in \mathbb{R}$, this is satisfied and V is convex. Similarly, if $\{p_m\}$ is a sequence of polynomials in $V$and $\lim_{m\to\infty} p_m = p$ exists, then p is a polynomial of degree at most n, and for any $t \in \mathbb{R}$, we have $0 \leq \lim_{m\to\infty} p_m(t) = p(t)$. Thus, $p \in V$, and V is closed.

2) Let K be the triangle with vertices $(0,0), (0,1)$ and $(1,0)$, and let L be the reflection of K to the origin. Determine the sets $K + L$ and $K - L$.
(5 points)

Solution: Note that $L = -K$. Furthermore, $K + K$ and $K - K$ can be obtained as $\bigcup_{x\in K}(x + K)$ and $\bigcup_{x\in K}(x - K)$. From this, we have that $K - L = K + K$ is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$, and $K + L = K - K$ is the convex hexagon with vertices $(1,0), (0,1), (-1,1), (-1,0), (0,-1)$ and $(1,-1)$.

3) Show that if $S \subseteq \mathbb{R}^2$ is an arbitrary nonempty set and $p \in \operatorname{int}\operatorname{conv} S$, then there are points $p_1, p_2, p_3, p_4 \in S$ such that $p \in \operatorname{int}\operatorname{conv}\{p_1, p_2, p_3, p_4\}$.
(5 points)

Solution: Since $p \in \operatorname{int}\operatorname{conv} S$, there are points $q_1, q_2, q_3 \in \operatorname{conv} S$ such that p is in the interior of the triangle $\operatorname{conv}\{q_1, q_2, q_3\}$. By Carathéodory's theorem, for each q_i there are at most 3 points of S whose convex hull contains q_i. Thus, there are at most nine points of S whose convex hull P contains p in its interior. Then P is a convex k-gon with $k \leq 9$. Let us triangulate P with all diagonals starting at a given vertex. Since p lies on at most one of these diagonals, p lies in the interior of the union of two consecutive triangles.

4) Prove that if $K \subset \mathbb{R}^n$ is convex, then $K + K = 2K$. Give an example showing that the same property does not hold if we drop the condition that K is convex.
(5 points)

Solution: Since $K + K = \{p + q : p, q \in K$ and $2K = \{2p : p \in K$, we have $2K \subseteq K + K$ for all sets K. Now, let K be convex and let $p, q \in K$. Then $\frac{p+q}{2} \in K$ by convexity, which implies that $2 \cdot \frac{p+q}{2} = p + q \in 2K$. This implies that $K + K \subseteq 2K$, and thus, $K + K = 2K$. Finally, if $K = \{0, 1\} \subset \mathbb{R}$, then $K + K = \{0, 1, 2\}$ has 3 elements whereas $2K = \{0, 2\}$ has two elements, and thus, $K + K \neq 2K$.