Convex Geometry

2nd midterm - SAMPLE

1) A compact, convex set $K \subset \mathbb{R}^n$ is called *strictly convex* if its boundary does not contain a segment. Show that K is strictly convex if and only if every boundary point of K is an extremal point. (5 points)

Solution:
If the boundary of K contains a segment $[q, r]$, then the relative interior points of $[q, r]$ are not extremal points of K. On the other hand, assume that a boundary point p is not an extremal point of K. Then there is a segment $[q, r] \subseteq K$ containing p in its relative interior. Then $q, r \in \mathrm{bd}(K)$, since if one of them, say $q \in \mathrm{int}(K)$, then the fact that $r \in K$ would imply that any point of $[q, r] \setminus \{r\}$ lies in the interior of K, which would contradict the fact that $p \in \mathrm{bd}(K)$. Using the same argument it also follows that any point of $[p, q]$ lies in $\mathrm{bd}(K)$. Thus, $\mathrm{bd}(K)$ contains a segment.

2) Can the given sets be separated by a line? If yes, find a separating line. (5 points)

$$A = \{(-1, 1), (0, -1), (4, 1)\}, \qquad B = \{(-3, 1), (1, -2)\}$$

Solution:
The sets A and B can be separated by a line if and only if $\mathrm{conv}(A)$ and $\mathrm{conv}(B)$ can be separated by a line. Furthermore, since $\mathrm{int}\,\mathrm{conv}(A) \neq \emptyset = \mathrm{int}\,\mathrm{conv}(B)$, by Theorem 1 of Lecture 6, $\mathrm{conv}(A)$ and $\mathrm{conv}(B)$ can be separated by a line if and only if $\mathrm{int}\,\mathrm{conv}(A) \cap \mathrm{conv}(B) = \emptyset$. We show it by checking that A is contained in the open half plane 'above' the affine hull of B (i.e. the line through the two points of B); this will also imply that $\mathrm{aff}(B)$ separates A and B. The equation of the line containing B is $y = -\frac{3}{4}x - \frac{5}{4}$. Substituting the x-coordinates of the points $(-1, 1), (0, -1), (4, 1)$ into this equation, we obtain the values $-\frac{1}{2}, -\frac{5}{4}, -\frac{17}{4}$, respectively. Since each of these values is strictly less than the y-coordinate of the corresponding point, we have that A is contained in the open half plane 'above' the affine hull of B. Thus, A and B can be separated by a line, and an example of a separating line is the one with equation $y = -\frac{3}{4}x - \frac{5}{4}$.

3) Show that if $K_1, K_2, \ldots, K_m \subseteq \mathbb{R}^n$ are closed, convex sets and $\bigcap_{i=1}^m K_i \neq \emptyset$, then $\chi\left(\bigcup_{i=1}^m K_i\right) = 1$. (5 points)

Solution:

By the Exclusion-Inclusion formula for Euler characteristic, we have

$$\chi(K_1 \cup K_2 \cup \ldots \cup K_m) = \sum_{j=1}^{m} (-1)^{j-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq k} \chi(K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_j}).$$

As $\bigcap_{i=1}^m K_i \neq \emptyset$, no intersection in the above formula is $\emptyset$, and since the Euler characteristic of a closed convex set is one, and the intersection of closed, convex sets is closed and convex, we have that

$$\chi(K_1 \cup K_2 \cup \ldots \cup K_m) = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \ldots + (-1)^m \binom{m}{m} =$$

$$= \binom{m}{0} - \sum_{j=0}^{m} (-1)^j \binom{m}{j} = \binom{m}{0} = 1.$$

4) Using Euler's theorem prove that the coordinates of the f-vector $f = (f_0, f_1, f_2, 1)$ of a 3-dimensional convex polytope satisfy the inequalities:

$$\frac{f_0}{2} + 2 \leq f_2 \leq 2f_0 - 4; \quad \frac{3f_0}{2} \leq f_1 \leq 3f_0 - 6. \text{ (5 points)}$$

Solution:

Euler's theorem states that for any 3-dimensional convex polytope P with f-vector $f = (f_0, f_1, f_2, 1)$, we have $f_0 + f_1 + f_2 = 2$. On the other hand, since every vertex of P lies on at least 3 edges, and every edge contains exactly two vertices, we have $2f_1 \geq 3f_0$, implying that $f_1 \geq \frac{3f_0}{2}$ and also that $2(f_0 + f_2 - 2) \geq 3f_0$, or equivalently that $\frac{f_0}{2} + 2 \leq f_2$. Similarly, every face of P has at least 3 edges, and every edge belongs to exactly 2 faces, and hence, we have $2f_1 \geq 3f_2$. Thus, $2f_1 \geq 3(f_1 + 2 - f_0)$, which implies that $f_1 \leq 3f_0 - 6$, and $2(f_0 + f_2 - 2) \geq 3f_2$, which implies that $f_2 \leq 2f_0 - 4$.