LECTURE 10: EULER'S THEOREM FOR POLYTOPES

This week we investigate the properties of the Euler characteristics of convex polytopes.

Lemma 1. *Let $P \subset \mathbb{R}^n$ be an n-dimensional (convex) polytope. Then*

$$\chi(\operatorname{bd} P) = 1 + (-1)^{n-1}, \quad \textit{and} \quad \chi(\operatorname{int} P) = (-1)^n.$$

Proof. By Corollary 2 of the 8th lecture $\operatorname{bd} P$ is the union of the facets of P, and thus, by Lemma 3 of the 7th lecture $I[\operatorname{bd} P] \in \mathcal{K}(\mathbb{R}^n)$ and thus, $\chi(\operatorname{bd} P)$ exists. We prove the first equality by induction.

If $n = 1$, then P is a closed segment, for which the assertion readily follows. Assume that P is an n-dimensional polytope, and also that the statement holds for $(n-1)$-dimensional polytopes. We use the notation of Lemma 1 of the 7th lecture. By the lemma,

$$\chi(\operatorname{bd} P) = \sum_{t \in \mathbb{R}} \left(\chi(H_t \cap \operatorname{bd} P) - \lim_{\varepsilon \to 0^+} \chi(H_{t-\varepsilon} \cap \operatorname{bd} P) \right).$$

Let $t_{\min} = \min_{x \in P} x_n$ and $t_{\max} = \max_{x \in P} x_n$, where $x = (x_1, \ldots, x_n)$. Then, for every $t_{\min} < t < t_{\max}$, the set $P \cap H_t$ is an $(n-1)$-dimensional polytope, and thus, by the induction hypothesis. $\chi(H_t \cap \operatorname{bd} P) = \chi(\operatorname{bd}(H_t \cap P)) = 1 + (-1)^{n-2}$. If $t = t_{\min}$ or $t = t_{\max}$, then $H_t \cap \operatorname{bd} P$ is a face of the polytope, and thus, $\chi(H_t \cap \operatorname{bd} P) = 1$. Furthermore, if $t > t_{\max}$ or $t < t_{\min}$, then $\chi(H_t \cap \operatorname{bd} P) = 0$. Summing up:

$$\chi(\operatorname{bd} P) = 1 - (1 + (-1)^{n-2}) + 1 = 1 + (-1)^{n-1}.$$

Finally, by $I[\operatorname{int} P] = I[P] - I[\operatorname{bd} P]$, we have

$$\chi(\operatorname{int} P) = 1 - (1 + (-1)^{n-1}) = (-1)^n.$$

□

Definition 1. *Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope. If $i = 0, 1, \ldots, n-1$, let $f_i(P)$ denote the number of the i-dimensional faces of P. Then the vector $f(P) = (f_0(P), f_1(P), \ldots, f_{n-1}(P), 1) \in \mathbb{R}^{n+1}$ is called the f-*vector *of P.*

We remark that the last coordinate is the consequence of the convention, often appearing in the literature, which regards P as an n-dimensional face of itself.

To prove our next theorem we need a lemma, with respect to which we should clarify that the relative interiors of singletons (i.e. 0-dimensional affine subspaces) are themselves.

Lemma 2. *Let $P \subset \mathbb{R}^n$ be an n-dimensional polytópe and let $x \in \mathrm{bd}(P)$ be arbitrary. Then there is a unique face of P containing x in its relative interior.*

Proof. Let F be the intersection of the faces containing x. Since P has only finitely many faces, and the intersection of finitely many faces is a face, it follows that F is a face of P. As $x \in F$, therefore F is a proper face. We show that $x \in \mathrm{relint}(F)$, and that F is the only face of P with this property.

Assume that $x \in \mathrm{relbd}(F)$. Since F is a convex polytope, F has a face F' containing x. But then Proposition 3 of the 8th lecture implies that F' is a proper face of P, and thus we have found a face F' containing x with $F \not\subseteq F'$, which contradicts the definition of F. Thus, $x \in \mathrm{relint}(F)$.

For contradiction, let $F' \neq F$ be a proper face of P satisfying $x \in \mathrm{relint}(F')$. Then, by the definition of F, we have $F \subset F'$. On the other hand, since F is a face of P, there is a hyperplane H supporting P with $H \cap P = F$. This hyperplane supports also the convex polytope F' in F, implying that F is a proper face of F'. Thus, $x \in F \subset \mathrm{relbd}(F')$; a contradiction. □

Theorem 1 (Euler)**.** *Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope. Then*

$$\sum_{i=0}^{n-1} (-1)^i f_i(P) = 1 + (-1)^{n-1}.$$

Proof. Observe that $I[P] = \sum I[\mathrm{relint} F]$, where the summation is taken over all nonempty faces of P, and P itself. Applying the valuation χ to both sides of this equation, the statement follows from the previous lemma. □

From now on, we denote by $B_r(x)$ the closed ball of radius r and center x.

The main concept of this lecture is the following.

Definition 2. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set. Then the* polar of A *is the set*

$$A^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \textit{ for every } x \in A\}.$$

Példák.

(1) $\{o\}^* = \mathbb{R}^n$,

(2) If $x \neq o$, then $\{x\}^*$ is the closed half space, containing o, whose boundary is perpendicular to x and its distance from o is $\frac{1}{||x||}$.
(3) For any $r > 0$, $B_r(o)^* = B_{\frac{1}{r}}(o)$. This statement readily follows from the previous example, since the intersection of the closed half spaces, containing o, whose distance from o is $\frac{1}{r}$ coincides with $B_{\frac{1}{r}}(o)$.

The next theorem summarizes some simple properties of polarity.

Theorem 2. (a) *For any set $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$, we have $A^* = \bigcap_{a\in A}\{a\}^*$.*
(b) *For any nonempty sets $A_i \subseteq \mathbb{R}^n$, $i \in I$, we have $\left(\bigcup_{i\in I} A_i\right)^* = \bigcap_{i\in I} A_i^*$.*
(c) *For any $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$, the set A^* is a closed, convex set containing o.*
(d) *If $A_1 \subseteq A_2 \subseteq \mathbb{R}^n$ are nonempty, then $A_2^* \subseteq A_1^*$.*
(e) *If $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$ and $\lambda > 0$, then $(\lambda A)^* = \frac{1}{\lambda}A^*$.*

Proof. Part (a) of the theorem is a direct consequence of the definition. Part (b) can be shown similarly, since

$$\left(\bigcup_{i\in I} A_i\right)^* = \bigcap_{x\in\bigcup_{i\in I} A_i} \{x\}^* = \bigcap_{i\in I}\left(\bigcap_{x\in A_i}\{x\}^*\right) = \bigcap_{i\in I} A_i^*.$$

To prove part (c) consider the fact that for any $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$, the set A^* is either $\mathbb{R}^n$ (which is a closed, convex set containing o), or the intersection of closed half spaces containing o. Since closed half spaces are convex sets, and the intersection of closed, convex sets containing o is a closed, convex set containing o, (c) follows. Part (d) is a consequence of part (a). Finally, if $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$ and $\lambda > 0$, then

$$(\lambda A)^* = \{y \in \mathbb{R}^n : \langle \lambda x, y\rangle \leq 1 \text{ for every } x \in A\} = \{y \in \mathbb{R}^n : \langle x, \lambda y\rangle \leq 1 \text{ for every } x \in A\} =$$

$$= \left\{\frac{1}{\lambda}z \in \mathbb{R}^n : \langle x, z\rangle \leq 1 \text{ for every } x \in A\right\} = \frac{1}{\lambda}\{z \in \mathbb{R}^n : \langle x, z\rangle \leq 1 \text{ for every } x \in A\} = \frac{1}{\lambda}A^*.$$

□

The next two statements investigate the polars of special classes of sets.

Proposition 1. *Let $K \subset \mathbb{R}^n$ be a compact, convex set containing o in its interior. Then K^* is a compact, convex set containing o in its interior.*

Proof. By part (c) of Theorem 2, K^* is a closed, convex set containing o. We show that K^* is bounded and it contains o in its interior. According to our conditions, there are constants $0 < r < R$ such that $B_r(o) \subseteq K \subseteq B_R(o)$. From this, by part (d) of Theorem 2 it follows that

$$B_{\frac{1}{R}}(o) = B_R(o)^* \subseteq K^* \subseteq B_r(o)^* = B_{\frac{1}{r}}(o),$$

which yields the statement. □

Proposition 2. *Let $K \subseteq \mathbb{R}^n$, $K \neq \emptyset$. Then $(K^*)^* = K$ holds if and only if K is closed, convex, and $o \in K$.*

Proof. If $(K^*)^* = K$, then by part (c) of Theorem 2, K is closed, convex and $o \in K$. We assume that K is closed, convex and $o \in K$, and show that $(K^*)^* = K$. By the definition of polar, for every $x \in K$ and $y \in K^*$, we have $\langle x, y \rangle \leq 1$, and thus, $K \subseteq (K^*)^*$. Now, let $x \notin K$ be arbitrary. Since K is closed and convex, by Theorem 6 of the 4th lecture there is a hyperplane H that strictly separates x and K. Let H^+ denote the closed half space bounded by H and containing $o \in K$. By the example in the beginning of the lecture, the half space H^+ is the polar of the set $\{y\}$, where the distance of H from o is $\frac{1}{|y|}$, and y is an outer normal of H^+. But then $x \notin \{y\}^*$ yields $\langle x, y \rangle > 1$, and $K \subset \{y\}^*$ yields $\langle z, y \rangle \leq 1$ for every $z \in K$. Thus, in this case $y \in K^*$, implying $x \notin (K^*)^*$. This yields $(K^*)^* \subseteq K$, which implies the assertion. □

The main result of this lecture is as follows.

Theorem 3. *Let $K \subset \mathbb{R}^n$ be a compact, convex set containing o in its interior. To any proper face F of K assign the set*

$$F^\circ = \{y \in K^* : \langle x, y \rangle = 1 \textit{ for every } x \in F\}.$$

Then F° is a proper face of K^*, and the map $F \mapsto F^\circ$ is a bijection between the proper faces of K and K^* that reverses containment relation.

Proof. Let $H = \{y \in \mathbb{R}^n : \langle v_0, y \rangle = 1\}$ be an arbitrary supporting hyperplane of K satisfying $F = H \cap K$. Since $\langle v_0, y \rangle \leq 1$ for every $y \in K$ and $\langle v_0, y \rangle = 1$ for every $y \in F$, we have $v_0 \in F^\circ$. Thus, $F^\circ \neq \emptyset$. Now, let $x_0 \in \text{relint}(F)$ and $H' = \{y \in \mathbb{R}^n : \langle y, x_0 \rangle = 1\}$. By the definition of polar set and $v_0 \in H'$, we have that H' is a supporting hyperplane of K^*, implying that $F' = K^* \cap H'$ is a proper face of K°. We show that $F' = F^\circ$.

By the definition of F°, $F^\circ \subset H'$ holds, and thus, $F^\circ \subseteq F'$. Now, let $y_0 \in K^* \setminus F^\circ$. Then, there is some $z \in F$ such that $\langle z, y_0 \rangle < 1$. As $x_0 \in \text{relint}(F)$, there is a segment $[z, w] \subseteq F$ with $x_0 \neq w$. Then x_0

can be written in the form $x_0 = tz + (1-t)w$ for some $t \in (0,1]$. But $w \in F$ and $y_0 \in K^*$ imply $\langle w, y_0 \rangle \leq 1$, from which

$$\langle x_0, y_0 \rangle t \langle z, y_0 \rangle + (1-t)\langle w, y_0 \rangle < 1,$$

that is, $y_0 \notin F'$. Thus, we have shown that $F^\circ = F'$ yielding, in particular, that $F \mapsto F^\circ$ is a face of K^*.

We continue the proof from here next week. $\square$